

SPECTRAL GAP PROPERTY FOR RANDOM DYNAMICS ON THE REAL LINE AND MULTIFRACTAL ANALYSIS OF GENERALISED TAKAGI FUNCTIONS

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ABSTRACT. We consider the random iteration of finitely many expanding $\mathcal{C}^{1+\varepsilon}$ diffeomorphisms on the real line without a common fixed point. We derive the spectral gap property of the associated transition operator acting on Hölder spaces. As an application we introduce generalised Takagi functions on the real line and we perform a complete multifractal analysis of the pointwise Hölder exponents of these functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamical systems on the real line. The theory of dynamical systems is used to describe various subjects in basically all areas of natural and social sciences. Since nature and any other environment have a lot of random terms, it is very natural and important not only to consider the dynamics of iteration of one map, but also to consider random dynamics. Many researchers in various fields have found and investigated many kinds of new phenomena in random dynamics which cannot hold in deterministic dynamics. These phenomena arise from the effect of randomness or noise and they are called *randomness-induced phenomena* or *noise-induced phenomena* ([JS15, JS17, Sum11, Sum13]). Under certain conditions, because of the effect of randomness or noise, the chaoticity of the system becomes milder, but the system still has some complexity. Hence regarding such random dynamical systems, our aim is to investigate the *gradation between chaos and order*.

To find and to study quantities describing the gradation between chaos and order, we combine ideas of random dynamical systems, ergodic theory (in particular, thermodynamic formalism), iterated function systems, and fractal geometry. More precisely, for any random dynamical system in our setting, there exists an exponent $\alpha_- \in (0, 1)$ such that for each α with $0 < \alpha < \alpha_-$, the transition operator of the system behaves well (e.g., it has a spectral gap property) on the space \mathcal{C}^α of α -Hölder continuous functions endowed with α -Hölder norm, but for each α with $\alpha_- < \alpha < 1$, the transition operator of the system does not behave well (Theorem 1.1, Corollary 1.5). This quantity α_- describes the gradation between chaos and order for the system. Furthermore, to provide a refined gradation, we investigate the pointwise Hölder exponents of the limit state functions (i.e., fixed points of the transition operator) and their (higher order) partial derivatives with respect to the probability parameters. It turns out that the pointwise Hölder exponents have a complicated fine structure which can be suitably investigated using the multifractal analysis and the concept of fractal dimension (Theorems 1.3, 1.4). The objects appearing in the multifractal analysis also describe the gradation between chaos and order for the system. Moreover, we present a new general framework to study a large class of fractal functions. In particular, we shed new light on the regularity properties of the classical Takagi function in our framework (see Theorem 1.4 and Proposition 1.6).

Throughout, let $I := \{1, \dots, s+1\}$, $s \geq 1$, and let $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in I$, be a family of $\mathcal{C}^{1+\varepsilon}$ diffeomorphisms with ε -Hölder continuous derivatives for some $\varepsilon > 0$. We say that $(f_i)_{i \in I}$ is *expanding* if there exists $\lambda > 1$

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such that $f_i'(x) \geq \lambda > 1$, for all $x \in \mathbb{R}$ and $i \in I$. The family $(f_i)_{i \in I}$ has *no common fixed point* if there exists no $x \in \mathbb{R}$ such that $f_i(x) = x$ for all $i \in I$.

We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the two-point compactification of \mathbb{R} endowed with a metric d on $\overline{\mathbb{R}}$ which is strongly equivalent to the Euclidean metric on compact subsets of \mathbb{R} , that is, for each compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that $C^{-1}|x - y| \leq d(x, y) \leq C|x - y|$, for all $x, y \in K$. For $i \in I$ we extend the definition of f_i from \mathbb{R} to $\overline{\mathbb{R}}$ by setting $f_i(\pm\infty) := \pm\infty$. We say that $(f_i)_{i \in I}$ is *contracting near infinity* if there exist neighborhoods V^\pm of $\pm\infty$ such that $\text{Lip}(f_i|_{V^\pm}) < 1$, $i \in I$. Here, for $D \subset \overline{\mathbb{R}}$ and $g : D \rightarrow \mathbb{R}$, we have set $\text{Lip}(g) := \sup_{x, y \in D, x \neq y} d(g(x), g(y)) / d(x, y)$. Note that if $(f_i)_{i \in I}$ is contracting near infinity then $\text{Lip}(f_i) < \infty$ for each $i \in I$. We refer to section 7 for details about the property of contraction near infinity.

Throughout, we assume that $(f_i)_{i \in I}$ is expanding, has no common fixed point, and is contracting near infinity. For $\mathbf{p} = (p_1, \dots, p_s) \in (0, 1)^s$ with $\sum_{i=1}^s p_i < 1$, let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. Let $\mathcal{C}(\overline{\mathbb{R}})$ denote the Banach space endowed with the supremum norm $\|\cdot\|_\infty$. Define the transition operator

$$M_{\mathbf{p}} : \mathcal{C}(\overline{\mathbb{R}}) \rightarrow \mathcal{C}(\overline{\mathbb{R}}), \quad M_{\mathbf{p}}h = \sum_{i \in I} p_i \cdot h \circ f_i, \quad h \in \mathcal{C}(\overline{\mathbb{R}}).$$

For $\alpha > 0$ let $\mathcal{C}^\alpha(\overline{\mathbb{R}})$ denote the Banach space of α -Hölder continuous functions (see Section 2). Note that $M_{\mathbf{p}}(\mathcal{C}^\alpha(\overline{\mathbb{R}})) \subset \mathcal{C}^\alpha(\overline{\mathbb{R}})$. To state our first main result we say that $M_{\mathbf{p}}$ has the *spectral gap property* if its spectrum consists of finitely many eigenvalues of modulus one, and the rest of the spectrum is contained in a ball of radius strictly less than one. We say that $(f_i)_{i \in I}$ satisfies the *separating condition* if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$, we have $f_i^{-1}(O) \cap f_j^{-1}(O) = \emptyset$. For the definition of the bottom of the spectrum $\alpha_- = \alpha_-(\mathbf{p})$ we refer to Section 2.2. For $\mathbf{a} \in \mathbb{R}^s$ and $\delta > 0$ we denote by $B(\mathbf{a}, \delta) \subset \mathbb{R}^s$ the open ball of radius δ with center \mathbf{a} in \mathbb{R}^s .

Theorem 1.1 (Theorem 2.4 and Theorem 2.15). *For every $\mathbf{p}_0 \in (0, 1)^s$ there exist $\delta > 0$ and $\alpha > 0$ such that $M_{\mathbf{p}} : \mathcal{C}^\alpha(\overline{\mathbb{R}}) \rightarrow \mathcal{C}^\alpha(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. If $(f_i)_{i \in I}$ satisfies the separating condition, then the previous assertion holds for any $\alpha < \alpha_-(\mathbf{p}_0)$.*

By combining with the perturbation theory of linear operators we can derive that the probability of tending to infinity $T := T_{\mathbf{p}} : \overline{\mathbb{R}} \rightarrow [0, 1]$ (see (2.1) below for the definition) depends real analytically on \mathbf{p} (see Theorem 2.4 for the detailed statement). This allows us to make the following definition.

Definition 1.2. We denote by $\mathcal{T} := \mathcal{T}_{\mathbf{p}}$ the \mathbb{R} -vector space of generalised Takagi functions generated by

$$C_{\mathbf{n}}(x) := C_{\mathbf{n}, \mathbf{p}}(x) := \frac{\partial^{\sum_{i=1}^s n_i}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_s^{n_s}} T_{(u_1, \dots, u_s)}(x) \Big|_{(u_1, \dots, u_s) = \mathbf{p}}, \quad \mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s, \quad x \in \overline{\mathbb{R}}.$$

For the reason why we call the elements of \mathcal{T} generalised Takagi functions, we refer to Remark 2.17. We then proceed to investigate the regularity of elements of \mathcal{T} . The pointwise Hölder exponent of $C \in \mathcal{T}$ at $x \in \mathbb{R}$ is denoted by $\text{Höl}(C, x)$ (see (3.2) below for the definition). We denote by J the Julia set of $(f_i)_{i \in I}$ (see Section 2.1). For $\mathbf{p} \in (0, 1)^s$ we define $\alpha_+ = \alpha_+(\mathbf{p})$ in Section 2.2. We say that $(f_i)_{i \in I}$ satisfies the *open set condition* if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$ we have $f_i^{-1}(O) \cap f_j^{-1}(O) = \emptyset$. By t^* we denote the Legendre transform of the function t defined implicitly by a certain topological pressure functional (see Section 3.2).

Theorem 1.3 (Theorem 3.14). *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C \in \mathcal{T} \setminus \{0\}$. Then we have for all $\alpha \in [\alpha_-, \alpha_+]$,*

$$\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} = -t^*(-\alpha),$$

and for $\alpha \notin [\alpha_-, \alpha_+]$ we have $\{x \in J \mid \text{Höl}(C, x) = \alpha\} = \emptyset$.

We prove the following result regarding the global Hölder continuity of elements of \mathcal{T} .

Theorem 1.4 (Theorem 4.1, Corollary 4.3). *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Then, for each $C \in \mathcal{T} \setminus \{0\}$ we have $\alpha_- = \sup \{ \alpha \geq 0 \mid C \in \mathcal{C}^\alpha(\overline{\mathbb{R}}) \}$. Further, $T \in \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$.*

Corollary 1.5 (Corollary 4.4). *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. If $\alpha_- < 1$ then $\lim_{n \rightarrow \infty} \|M_{\mathbf{p}}^n\|_\alpha = \infty$, for each $\alpha_- < \alpha < 1$, where $\|M_{\mathbf{p}}^n\|_\alpha$ denotes the operator norm of $M_{\mathbf{p}}^n$ on $\mathcal{C}^\alpha(\overline{\mathbb{R}})$.*

Regarding the existence of points of non-differentiability of elements of \mathcal{T} we prove the following. Let $\mathbf{e}_k \in \mathbb{N}^s$ denote the k -th unit vector in \mathbb{N}_0^s , $1 \leq k \leq s$. We use C_m to denote $C_{(m)}$ for $m \in \mathbb{N}_0$.

Proposition 1.6 (Proposition 5.1). *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition.*

- (1) *If $\alpha_- < 1$, then there exists a dense subset $E \subset J$ of positive Hausdorff dimension such that, for every $C \in \mathcal{T} \setminus \{0\}$ and every $x \in E$, C is not differentiable at x .*
- (2) *If $\alpha_- = 1$, then $C_{\mathbf{e}_k}$ is nowhere differentiable on J , for $1 \leq k \leq s$. Moreover, if $s = 1$, then C_m is nowhere differentiable on J , for every $m \geq 1$.*

For applications of our results to conjugacies of interval maps we refer to Section 6. In fact, if $(f_i)_{i \in I}$ satisfies the open set condition, then the probability of tending to infinity $T_{\mathbf{p}}$ can also be characterised as the conjugacy map between the expanding dynamical system defined by $(f_i)_{i \in I}$ on J and the dynamical system given by the piecewise linear map on $[0, 1]$ with $(s + 1)$ full branches and slopes given by $(1/p_i)_{i \in I}$ (see Lemma 6.1). By the rigidity dichotomy in [JKPS09, Theorem 1.2], if J is an interval, then either $\alpha_-(\mathbf{p}) = \alpha_+(\mathbf{p})$ and $T_{\mathbf{p}}$ is a $\mathcal{C}^{1+\varepsilon}$ -diffeomorphism, or $\alpha_-(\mathbf{p}) < \alpha_+(\mathbf{p})$ and the set of non-differentiability points of $T_{\mathbf{p}}$ has positive Hausdorff dimension. For general families $(f_i)_{i \in I}$ satisfying the open set condition, we can show that $\alpha_-(\mathbf{p}) = \alpha_+(\mathbf{p})$ if and only if $T_{\mathbf{p}} \in \mathcal{C}^{\dim_H(J)}(\overline{\mathbb{R}})$ (see Section 6).

Higher order derivatives of the classical Takagi function have been considered in [AK06], where it is shown that the classical Takagi function and the higher order derivatives of the Lebesgue singular function for $p = 1/2$ are nowhere differentiable and convex Lipschitz ([MW86]). These results are covered by our general theory. Namely, since for this special case we have $\alpha_- = 1$ and $s = 1$, the non-differentiability follows from Proposition 1.6 (2). That these functions are α -Hölder continuous, for every $\alpha < 1$, follows from Theorem 1.4. In fact, we can also derive that the functions are convex Lipschitz (see Remark 4.2).

Generalised Takagi functions have also been introduced in [HY84]. As already observed in [AK06], the higher-order derivatives of a systems $(f_i)_{i \in I}$ with constant derivatives are not covered by the setting in [HY84]. In [SS91] it is shown that the Lebesgue singular function depends real analytically on the parameter, and its higher order derivatives are considered. We point out that our general definition of \mathcal{T} is a far-reaching generalisation of the above concept, where we consider an arbitrary finite number of $\mathcal{C}^{1+\varepsilon}$ diffeomorphisms and arbitrary linear combinations of higher order partial derivatives of the probability of tending to infinity with $s \geq 1$ parameters.

We remark that in the previous works of the authors [JS15, JS17] we dealt with the random complex dynamical systems satisfying the separating condition. However, in this paper, we deal with random dynamical systems on the real line satisfying the open set condition. Note that the separating condition implies the open set condition. When we deal with general systems satisfying the open set condition, we have to overcome new difficulties. In fact, the relation between the pointwise Hölder exponents of elements of \mathcal{T} and the corresponding dynamical quantities is much more involved than in the case of the separating condition. We developed several new ideas (see Propositions 3.9 and 3.13) to overcome these difficulties. In the case of the separating condition we show that α_- is the supremum of the exponents α for which the transition

operator has the spectral gap property on \mathcal{C}^α by developing some idea from [JS17] and providing a new approach.

In section 2, we derive the spectral gap property for the transition operator associated with random dynamical systems on the real line. In section 3, we perform a complete multifractal analysis of the pointwise Hölder exponents of the elements of \mathcal{T} associated with $(f_i)_{i \in I}$ and $\mathbf{p} \in (0, 1)^s$. In section 4, we investigate the global Hölder continuity of the elements of \mathcal{T} . In section 5, we study the (non-)differentiability of the elements of \mathcal{T} . In section 6, we show how our results are related to interval conjugacy maps. In section 7 (Appendix), we will show that, by modifying the $(f_i)_{i \in I}$ near infinity, we can always assume that an expanding family $(f_i)_{i \in I}$ is contracting near infinity with respect to a metric d which is strongly equivalent to the Euclidean metric on compact subsets of \mathbb{R} .

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2. SPECTRAL GAP PROPERTY

In this section we derive the spectral gap property for the transition operator associated with random dynamical systems on the real line induced by a family $(f_i)_{i \in I}$.

2.1. General results . Let $I^* := \bigcup_{n \in \mathbb{N}} I^n$. For $\omega \in I^*$ we denote by $|\omega|$ the unique $n \in \mathbb{N}$ such that $\omega \in I^n$. For $\omega = (\omega_1, \dots, \omega_n) \in I^n$ we let $f_{(\omega_1, \dots, \omega_n)} := f_{\omega_n} \circ \dots \circ f_{\omega_1}$. Also, for $\omega \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$ we put $\omega_n := (\omega_1, \dots, \omega_n) \in I^n$. Since $(f_i)_{i \in I}$ is contracting near infinity, there exist neighborhoods V^\pm of $\pm\infty$ in $\overline{\mathbb{R}}$ such that, for each $x \in V^+$ (resp. $x \in V^-$) we have for all $\omega \in I^{\mathbb{N}}$,

$$f_{\omega_n}(x) \rightarrow +\infty \quad (\text{resp. } f_{\omega_n}(x) \rightarrow -\infty), \quad \text{as } n \rightarrow \infty.$$

We put

$$V := V^+ \cup V^-.$$

Denote by $G := \langle f_1, \dots, f_{s+1} \rangle := \{f_\omega \mid \omega \in I^*\}$ the semigroup generated by f_1, \dots, f_{s+1} where the semigroup operation is the composition of functions. The *Julia set of G* is defined as

$$J := \{x \in \overline{\mathbb{R}} \mid G \text{ is not equicontinuous in any neighborhood of } x\}.$$

Note that the inverse maps $(f_i^{-1})|_{\overline{\mathbb{R}} \setminus V}$, $i \in I$, form a contracting conformal iterated function system (see e.g. [Fal03, MU03]) and J is the limit set (or attractor) of this system.

Let $\Sigma := I^{\mathbb{N}}$. For $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$ we define

$$J_\omega := \bigcap_{n \in \mathbb{N}} \left(f_{\omega_n} \right)^{-1} (\overline{\mathbb{R}} \setminus V).$$

Note that J_ω is a singleton because $(f_i)_{i \in I}$ is expanding. We define the coding map $\pi : \Sigma \rightarrow \mathbb{R}$ given by

$$\bigcap_{n \in \mathbb{N}} (f_{\omega_n})^{-1} (\overline{\mathbb{R}} \setminus V) = \{\pi(\omega)\}, \quad \omega \in \Sigma.$$

It is easy to see that

$$J = \bigcup_{\omega \in \Sigma} J_\omega = \pi(\Sigma).$$

The kernel Julia set of G ([Sum11]) is given by

$$J_{\ker} := \bigcap_{g \in G} g^{-1}(J) \subset J.$$

For $\mathbf{p} = (p_1, \dots, p_s) \in (0, 1)^s$ with $\sum_{i=1}^s p_i < 1$, let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. Let $\mu_{\mathbf{p}}$ denote the $(p_1, \dots, p_s, p_{s+1})$ Bernoulli measure on Σ .

Proposition 2.1. *We have $J_{\ker} = \emptyset$. Moreover, we have $\mu_{\mathbf{p}}(\{\omega \in \Sigma \mid x \in J_{\omega}\}) = 0$, for each $x \in \mathbb{R}$.*

Proof. Since $(f_i)_{i \in I}$ is expanding without a common fixed point, we have that, for all $x \in \mathbb{R}$ there exists $g_x \in G$ such that $g_x(x) \in V$. Hence, $J_{\ker} = \emptyset$. By [Sum11, Lemma 4.6] we have $\mu_{\mathbf{p}}(\{\omega \in \Sigma \mid x \in J_{\omega}\}) = 0$, for each $x \in \mathbb{R}$. \square

Recall that $M_{\mathbf{p}}$ is *almost periodic* if $(M_{\mathbf{p}}^n h)_{n \geq 1}$ is relatively compact in $\mathcal{C}(\overline{\mathbb{R}})$, for each $h \in \mathcal{C}(\overline{\mathbb{R}})$. The dual operator of $M_{\mathbf{p}}$ is given by $M_{\mathbf{p}}^* : \mathcal{M}_1(\overline{\mathbb{R}}) \rightarrow \mathcal{M}_1(\overline{\mathbb{R}})$, where $\mathcal{M}_1(\overline{\mathbb{R}})$ denotes the space of Borel probability measures on $\overline{\mathbb{R}}$ endowed with the topology of weak convergence. We define the compact subset

$$J_{\text{meas}} := \{m \in \mathcal{M}_1(\overline{\mathbb{R}}) \mid (M_{\mathbf{p}}^*)_{n \geq 1}^n \text{ is not equicontinuous in any neighbourhood of } m\} \subset \mathcal{M}_1(\overline{\mathbb{R}}).$$

The following fact is a special case of [Sum11, Proposition 4.7, Lemma 4.2(6)].

Proposition 2.2. *We have that $J_{\text{meas}} = \emptyset$ and that $M_{\mathbf{p}} : \mathcal{C}(\overline{\mathbb{R}}) \rightarrow \mathcal{C}(\overline{\mathbb{R}})$ is almost periodic.*

By a well-known result of Ljubich ([Lju83]) on almost periodic operators, we have

$$\mathcal{C}(\overline{\mathbb{R}}) = \{h \in \mathcal{C}(\overline{\mathbb{R}}) \mid \|M_{\mathbf{p}}^n h\|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty\} \oplus \overline{\text{span}\{h \in \mathcal{C}(\overline{\mathbb{R}}) \mid \exists \rho \in \mathbb{S}^1 \mid M_{\mathbf{p}} h = \rho h\}}.$$

As in [Sum11] we define the probability of tending to infinity

$$(2.1) \quad T_{\mathbf{p}} : \overline{\mathbb{R}} \rightarrow [0, 1], \quad T_{\mathbf{p}}(x) := \mu_{\mathbf{p}}\left\{\omega \in \Sigma \mid \lim_{n \rightarrow \infty} f_{\omega_n}(x) = \infty\right\}.$$

It follows from Proposition 2.1 that for every bounded measurable function h ,

$$(2.2) \quad \lim_{n \rightarrow \infty} M_{\mathbf{p}}^n h(x) = T_{\mathbf{p}}(x)h(\infty) + (1 - T_{\mathbf{p}}(x))h(-\infty),$$

where we note that $M_{\mathbf{p}}$ can also be defined on the space of bounded Borel measurable functions on $\overline{\mathbb{R}}$. Hence,

$$\text{span}\{h \in \mathcal{C}(\overline{\mathbb{R}}) \mid \exists \rho \in \mathbb{S}^1 \mid M_{\mathbf{p}} h = \rho h\} = \mathbb{R}T_{\mathbf{p}} \oplus \mathbb{R}1$$

and we have by Ljubich's result,

$$\|M_{\mathbf{p}}^n h - (h(\infty) - h(-\infty))T_{\mathbf{p}} - h(-\infty)1\|_{\infty} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

For $\alpha > 0$ we say that $h : \overline{\mathbb{R}} \rightarrow \mathbb{R}$ is α -Hölder continuous if

$$V_{\alpha} := \sup_{x \neq y} \left\{ \frac{|h(x) - h(y)|}{d(x, y)^{\alpha}} \right\} < \infty.$$

We say that a function is Hölder continuous if it is α -Hölder continuous for some $\alpha > 0$. We denote by $\mathcal{C}^{\alpha}(\overline{\mathbb{R}})$ the Banach space of α -Hölder continuous maps on $\overline{\mathbb{R}}$ endowed with the α -Hölder norm

$$\|h\|_{\alpha} := V_{\alpha} + \|h\|_{\infty}, \quad h \in \mathcal{C}^{\alpha}(\overline{\mathbb{R}}).$$

The following lemma is the key to derive the spectral gap property of $M_{\mathbf{p}}$ on $\mathcal{C}^{\alpha}(\overline{\mathbb{R}})$. The proof is inspired by [Sum13]. For $n \in \mathbb{N}$ and $\omega \in I^n$ we will use the notation

$$p_{\omega} := p_{\omega_1} \cdots p_{\omega_n}.$$

Hence, we have for $n \in \mathbb{N}$ and $h \in \mathcal{C}(\overline{\mathbb{R}})$

$$M_{\mathbf{p}}^n h = \sum_{\omega \in I^n} p_{\omega} \cdot (h \circ f_{\omega}) = \int h \circ f_{\omega_n} d\mu_{\mathbf{p}}(\omega).$$

Lemma 2.3. *For every $\mathbf{p}_0 \in (0, 1)^s$ there exist $\delta > 0$, $\alpha > 0$, $n \in \mathbb{N}$ and constants $0 < c < 1$ and $C > 0$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ and for every $h \in \mathcal{C}^{\alpha}(\overline{\mathbb{R}})$,*

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq (c\|h\|_{\alpha} + C\|h\|_{\infty})d(x, y)^{\alpha}, \quad x, y \in \overline{\mathbb{R}}.$$

Proof. Recall that $V = V^+ \cup V^-$. Since $(f_i)_{i \in I}$ is expanding without a common fixed point, we have that, for all $x \in \mathbb{R}$ there exists $g_x \in G$ and a compact neighborhood U_x of x in $\overline{\mathbb{R}}$ such that $g_x(U_x) \subset V$. Since $\overline{\mathbb{R}}$ is compact, there exist $t \in \mathbb{N}$ and $x_1, \dots, x_t \in \overline{\mathbb{R}}$ such that $\overline{\mathbb{R}} = \bigcup_{j=1}^t \text{Int}(U_{x_j})$, where $\text{Int}(A)$ denotes the interior of a set $A \subset \overline{\mathbb{R}}$. Since $\bigcup_{g \in G} g(V) \subset V$ we may assume that there exists $r \in \mathbb{N}$ such that, for each $j = 1, \dots, t$ there exists $\beta^j \in I^r$ with $g_{x_j} = f_{\beta^j}$. For $\omega \in I^*$ with $|\omega| = \ell$ we denote by $[\omega] := \{\tau \in \Sigma \mid \tau_1 = \omega_1, \dots, \tau_{\ell} = \omega_{\ell}\}$ the cylinder set of ω . Let

$$a := a(\mathbf{p}) := \max \{1 - \mu_{\mathbf{p}}([\beta^j]) \mid 1 \leq j \leq t\} < 1.$$

Recall that $\text{Lip}(g) < \infty$, for every $g \in G$. Let

$$\lambda := 2 \cdot \max \{\max \{\text{Lip}(f_{\omega}) \mid \omega \in I^r\}, 1\} \geq 2.$$

Let $R > 0$ be a Lebesgue number of the covering $(\text{Int}(U_{x_j}))_{1 \leq j \leq t}$ of $\overline{\mathbb{R}}$. Let $\alpha > 0$ such that

$$\eta := a\lambda^{\alpha} < 1.$$

Let $n \in \mathbb{N}$ to be determined later. Let $x, y \in \overline{\mathbb{R}}$. Since $M_{\mathbf{p}}$ is bounded, we may assume that $d(x, y) < R$. Let

$$n(x, y) := \max \left\{ k \geq 0 \mid \lambda^{-k} R > d(x, y) \right\}.$$

If $n(x, y) < n$, then $d(x, y) \geq \lambda^{-n} R$ and the desired estimate follows with $C := 2\lambda^{n\alpha} R^{-\alpha}$. Now, we consider the case $n(x, y) \geq n$. Then we have $d(x, y) < \lambda^{-n} R$. Consequently, for $j \leq n$ and $\omega \in I^{jr}$ we have $d(f_{\omega}(x), f_{\omega}(y)) \leq R$. By the definition of R there exists $i_0 \in \{1, \dots, t\}$ such that $B(x, R) \subset U_{x_{i_0}}$. Let $A(0) := [\beta^{i_0}] \subset \Sigma$ and $B(0) := \Sigma \setminus [\beta^{i_0}]$. We define inductively, for $j \geq 1$,

$$A(j) := \left\{ \omega \in B(j-1) \mid \exists i \in \{1, \dots, t\} \text{ such that } B(f_{\omega_{rj}}(x), R) \subset U_{x_i} \text{ and } (\omega_{rj+1}, \dots, \omega_{r(j+1)}) = \beta^i \right\}$$

and $B(j) := B(j-1) \setminus A(j)$. We have

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq \left| \sum_{j=0}^{n-1} \int_{A(j)} h(f_{\omega_{rn}}(x)) - h(f_{\omega_{rn}}(y)) d\mu_{\mathbf{p}}(\omega) \right| + \left| \int_{B(n-1)} h(f_{\omega_{rn}}(x)) - h(f_{\omega_{rn}}(y)) d\mu_{\mathbf{p}}(\omega) \right|.$$

Since $\mu_{\mathbf{p}}(A(j)) \leq a^j$ for every $j \leq n-1$, and $B(f_{\omega_{r(j+1)}}(x), R) \subset V$, for every $\omega \in A(j)$, we can estimate with

$$S := \max \{ \text{Lip}(f_{i|V^+}), \text{Lip}(f_{i|V^-}) \mid i \in I \} < 1 \quad \text{and} \quad \tilde{c} := \max \{ \eta, S^{r\alpha} \} < 1,$$

$$\begin{aligned} \int_{A(j)} |h(f_{\omega_{rn}}(x)) - h(f_{\omega_{rn}}(y))| d\mu_{\mathbf{p}}(\omega) &\leq a^j \sup_{\omega \in A(j)} |h(f_{\omega_{rn}}(x)) - h(f_{\omega_{rn}}(y))| \\ &\leq a^j \|h_{|V}\|_{\alpha} \sup_{\omega \in A(j)} \left\{ \left(S^{(n-j-1)r} d(f_{\omega_{r(j+1)}}(x), f_{\omega_{r(j+1)}}(y)) \right)^{\alpha} \right\} \\ &\leq a^j S^{(n-j-1)r\alpha} \lambda^{(j+1)\alpha} \|h_{|V}\|_{\alpha} d(x, y)^{\alpha} \\ &\leq \eta^j (S^{r\alpha})^{n-j-1} \lambda^{\alpha} \|h_{|V}\|_{\alpha} d(x, y)^{\alpha} = \tilde{c}^{n-1} \lambda^{\alpha} \|h_{|V}\|_{\alpha} d(x, y)^{\alpha}. \end{aligned}$$

Finally, we verify that

$$\begin{aligned} \int_{B(n-1)} \left| h(f_{\omega_{rn}}(x)) - h(f_{\omega_{rn}}(y)) \right| d\mu_{\mathbf{p}}(\omega) &\leq \mu(B(n-1)) \sup_{\omega \in B(n-1)} \left| h \circ f_{\omega_{rn}}(x) - h \circ f_{\omega_{rn}}(y) \right| \\ &\leq a^n \|h\|_{\alpha} \sup_{\omega \in B(n-1)} d\left(f_{\omega_{rn}}(x), f_{\omega_{rn}}(y)\right)^{\alpha} \\ &\leq a^n \|h\|_{\alpha} \lambda^{n\alpha} d(x, y)^{\alpha} \leq \eta^n \|h\|_{\alpha} d(x, y)^{\alpha}. \end{aligned}$$

We have thus shown that

$$\left| M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y) \right| \leq (\bar{c}^{n-1} \lambda^{\alpha} \|h\|_{\alpha} + \eta^n \|h\|_{\alpha}) d(x, y)^{\alpha}.$$

For n sufficiently large, the assertion of the lemma follows. It is clear that $a = a(\mathbf{p})$ and thus, α and the other constants involved, depend continuously on \mathbf{p} . Therefore, the assertion of the lemma holds with locally uniform constants. The proof is complete. \square

Theorem 2.4. *For every $\mathbf{p}_0 \in (0, 1)^s$ there exists $\delta > 0$ and $\alpha > 0$ such that $M_{\mathbf{p}} : \mathcal{C}^{\alpha}(\overline{\mathbb{R}}) \rightarrow \mathcal{C}^{\alpha}(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. In particular, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ we have $T_{\mathbf{p}} \in \mathcal{C}^{\alpha}(\overline{\mathbb{R}})$ and the convergence*

$$\|M_{\mathbf{p}}^n h - (h(\infty) - h(-\infty))T_{\mathbf{p}} - h(-\infty)1\|_{\alpha} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

is exponentially fast. Moreover, the map $\mathbf{p} \mapsto T_{\mathbf{p}} \in \mathcal{C}^{\alpha}(\overline{\mathbb{R}})$ is real-analytic on $B(\mathbf{p}_0, \delta)$.

Proof. By Lemma 2.3 there exist $\delta > 0$, $\alpha > 0$ and $n \in \mathbb{N}$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$, we have the Ionescu-Tulcea and Marinescu inequality

$$\|M_{\mathbf{p}}^n h\|_{\alpha} \leq c \|h\|_{\alpha} + (C+1) \|h\|_{\infty}.$$

Therefore, the theorem follows from the well-known result of [ITM50] in tandem with the perturbation theory for linear operators ([Kat76]). \square

Corollary 2.5. *For every $\mathbf{p}_0 \in (0, 1)^s$ there exists $\delta > 0$ and $\alpha > 0$ such that $\mathcal{T}_{\mathbf{p}} \subset \mathcal{C}^{\alpha}(\overline{\mathbb{R}})$ for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$.*

The next lemma can be proved exactly as in [JS17, Lemma 4.1].

Lemma 2.6. *For every $n \in \mathbb{N}_0^s$ we have*

$$C_{\mathbf{n}} = M_{\mathbf{p}} C_{\mathbf{n}} + \sum_{i=1}^s n_i (C_{\mathbf{n}-\mathbf{e}_i} \circ f_i - C_{\mathbf{n}-\mathbf{e}_i} \circ f_{s+1}).$$

The Bernoulli measure $\mu_{\mathbf{p}}$ on Σ defines the probability measure $\tilde{\mu}_{\mathbf{p}} = \mu_{\mathbf{p}} \circ \pi^{-1}$ on J with distribution function

$$F_{\mathbf{p}} : \overline{\mathbb{R}} \rightarrow [0, 1], \quad F_{\mathbf{p}}(x) = \tilde{\mu}_{\mathbf{p}}\{(-\infty, x]\}.$$

Lemma 2.7. *We have $M_{\mathbf{p}} F_{\mathbf{p}} = F_{\mathbf{p}}$.*

Proof. Clearly, $\mu_{\mathbf{p}}$ is a self-similar measure on Σ , i.e.,

$$\mu_{\mathbf{p}} = \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \sigma_i^{-1},$$

where $\sigma_i : \Sigma \rightarrow \Sigma$ is given by $\sigma_i(\omega) := i\omega$. Using that $f_i^{-1} \circ \pi = \pi \circ \sigma_i$, we obtain for every Borel set $B \subset \mathbb{R}$,

$$\begin{aligned} \tilde{\mu}_{\mathbf{p}}(B) &= \mu_{\mathbf{p}}(\pi^{-1}(B)) = \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \sigma_i^{-1}(\pi^{-1}(B)) \\ &= \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \pi^{-1}(f_i^{-1})^{-1}(B) = \sum_{i=1}^{s+1} p_i \tilde{\mu}_{\mathbf{p}}(f_i(B) \cap J). \end{aligned}$$

Setting $B := (-\infty, x]$ the lemma follows. \square

Lemma 2.8. $T_{\mathbf{p}}$ is the unique bounded Borel measurable function such that $M_{\mathbf{p}}T_{\mathbf{p}} = T_{\mathbf{p}}$ and $T_{\mathbf{p}|V^-} = 0$ and $T_{\mathbf{p}|V^+} = 1$. In particular, $T_{\mathbf{p}} = F_{\mathbf{p}}$.

Proof. By Lemma 2.7 we have that $M_{\mathbf{p}}T_{\mathbf{p}} = T_{\mathbf{p}}$. Clearly, $T_{\mathbf{p}}$ is bounded and measurable, and satisfies $T_{\mathbf{p}|V^-} = 0$ and $T_{\mathbf{p}|V^+} = 1$. Let h be another bounded Borel measurable function such that $M_{\mathbf{p}}h = h$ and $h|_{V^-} = 0$ and $h|_{V^+} = 1$. By (2.2) we have

$$h(x) = \lim_{n \rightarrow \infty} M_{\mathbf{p}}^n h(x) = h(\infty)T_{\mathbf{p}}(x) + h(-\infty)(1 - T_{\mathbf{p}}(x)) = T_{\mathbf{p}}(x),$$

which proves the asserted uniqueness. To prove that $T_{\mathbf{p}} = F_{\mathbf{p}}$ we note that $F_{\mathbf{p}}$ is bounded and Borel measurable, $M_{\mathbf{p}}F_{\mathbf{p}} = F_{\mathbf{p}}$, $F_{\mathbf{p}|V^-} = 0$ and $F_{\mathbf{p}|V^+} = 1$. The assertion of the lemma follows. \square

The following fact follows immediately from the definition of $F_{\mathbf{p}}$.

Fact 2.9. $F_{\mathbf{p}}$ (and hence, $T_{\mathbf{p}}$) is locally constant precisely on $\mathbb{R} \setminus J$.

2.2. Improved spectral gap property for systems with separating condition. In this section we derive an improved spectral gap property for systems satisfying the separating condition. We define the potentials

$$\varphi : \Sigma \rightarrow \mathbb{R}, \quad \varphi(\omega) := -\log |f'_{\omega_1}(\pi(\omega))|, \quad \text{and} \quad \psi := \psi_{\mathbf{p}} : \Sigma \rightarrow \mathbb{R}, \quad \psi(\omega) := \log p_{\omega}.$$

We define the shift map $\sigma : \Sigma \rightarrow \Sigma$, $\sigma((\omega_1, \omega_2, \dots)) := (\omega_2, \omega_3, \dots)$. For $u : \Sigma \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$ we denote by $S_n u := \sum_{k=0}^{n-1} u \circ \sigma^k$ the n th ergodic sum. Further we let

$$\alpha_- := \alpha_-(\mathbf{p}) := \inf_{\omega \in \Sigma} \liminf_{n \rightarrow \infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)}, \quad \alpha_+ := \alpha_+(\mathbf{p}) := \sup_{\omega \in \Sigma} \limsup_{n \rightarrow \infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)},$$

and we refer to α_- as the bottom of the spectrum.

Lemma 2.10. The map $\mathbf{p} \mapsto \alpha_-(\mathbf{p})$ is lower semi-continuous on $(0, 1)^s$.

Proof. Let $\mathbf{p}_0 \in (0, 1)^s$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\psi_{\mathbf{p}} - \psi_{\mathbf{p}_0}\| < \varepsilon / \min |\varphi|$ for every \mathbf{p} with $|\mathbf{p} - \mathbf{p}_0| < \delta$. Let \mathbf{p} with $|\mathbf{p} - \mathbf{p}_0| < \delta$, $\omega \in \Sigma$ and $(n_k) \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{S_{n_k} \psi_{\mathbf{p}}(\omega)}{S_{n_k} \varphi(\omega)} = \alpha_-(\mathbf{p}).$$

We have

$$\frac{S_{n_k} \psi_{\mathbf{p}}(\omega)}{S_{n_k} \varphi(\omega)} = \frac{S_{n_k} \psi_{\mathbf{p}_0}(\omega)}{S_{n_k} \varphi(\omega)} + \frac{S_{n_k} \psi_{\mathbf{p}}(\omega) - S_{n_k} \psi_{\mathbf{p}_0}(\omega)}{S_{n_k} \varphi(\omega)}.$$

Since

$$\left| \frac{S_{n_k} \psi_{\mathbf{p}}(\omega) - S_{n_k} \psi_{\mathbf{p}_0}(\omega)}{S_{n_k} \varphi(\omega)} \right| \leq \frac{n_k \|\psi_{\mathbf{p}} - \psi_{\mathbf{p}_0}\|}{n_k \min |\varphi|} < \varepsilon \quad \text{and} \quad \liminf_{k \rightarrow \infty} \frac{S_{n_k} \psi_{\mathbf{p}_0}(\omega)}{S_{n_k} \varphi(\omega)} \geq \alpha_-(\mathbf{p}_0),$$

we conclude that

$$\alpha_-(\mathbf{p}) = \lim_{k \rightarrow \infty} \frac{S_{n_k} \psi_{\mathbf{p}}(\omega)}{S_{n_k} \varphi(\omega)} \geq \liminf_{k \rightarrow \infty} \frac{S_{n_k} \psi_{\mathbf{p}_0}(\omega)}{S_{n_k} \varphi(\omega)} - \varepsilon \geq \alpha_-(\mathbf{p}_0) - \varepsilon.$$

□

Remark. Similarly, one can show that $\mathbf{p} \mapsto \alpha_+(\mathbf{p})$ is upper semi-continuous on $(0, 1)^s$.

Lemma 2.11. *There exists $\eta < 1$ such that for every compact set $K \subset \overline{\mathbb{R}} \setminus J$ there exists a constant $C_K < \infty$ such that for all $x, y \in K$ belonging to the same connected component of $\overline{\mathbb{R}} \setminus J$, we have for every $\omega \in I^*$,*

$$d(f_\omega(x), f_\omega(y)) \leq C_K \eta^{|\omega|} d(x, y).$$

Proof. There exists $N = N(K) \in \mathbb{N}$ such that for all $n \geq N$, $\omega \in I^n$ and all $x, y \in K$ belonging to the same component of $\overline{\mathbb{R}} \setminus J$, we have $f_\omega(x), f_\omega(y) \in V^+$ (resp. V^-). For otherwise, there exists $\tau \in I^{\mathbb{N}}$ such that $f_{\tau_n}(x) \in \overline{\mathbb{R}} \setminus V$, for all $n \in \mathbb{N}$, contradicting that $x \notin J$. Let $S := \max \{ \text{Lip}(f_i|_{V^+}), \text{Lip}(f_i|_{V^-}) \mid i \in I \}$. Since $(f_i)_{i \in I}$ is contracting near infinity, we have $S < 1$. For $n \geq N$ and $\omega \in I^n$ we have

$$d(f_\omega(x), f_\omega(y)) \leq S^{n-N} d(f_{\omega_N}(x), f_{\omega_N}(y)) \leq S^{n-N} \left(\max_{i \in I} \text{Lip}(f_i) \right)^N d(x, y) = S^n C_K d(x, y),$$

where we have set $C_K := S^{-N} (\max_{i \in I} \text{Lip}(f_i))^N$. The estimate for $n < N$ can be shown similarly. □

Definition 2.12. We say that $(f_i)_{i \in I}$ satisfies the *separating condition* if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$, we have $f_i^{-1}(\overline{O}) \cap f_j^{-1}(\overline{O}) = \emptyset$. If the separating condition holds, then we may always assume that O is bounded and that $J \subset O$.

The proof of the following lemma is standard and therefore omitted (see e.g., [MU03]).

Lemma 2.13 (Bounded distortion). *Let Ω be a bounded open set such that $f_i^{-1}(\Omega) \subset \Omega$ for $i \in I$. Then we have*

$$(2.3) \quad D := D(\Omega) := \sup \left\{ \frac{|f'_\gamma(v_1)|}{|f'_\gamma(v_2)|} \mid \gamma \in I^*, v_1, v_2 \in f_\gamma^{-1}(\overline{\Omega}) \right\} < \infty.$$

Lemma 2.14. *Suppose that $(f_i)_{i \in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0, 1)^s$ and $0 < \alpha < \alpha_-(\mathbf{p}_0)$. Then there exist $\delta > 0$, $n \in \mathbb{N}$ and constants $0 < c < 1$ and $C > 0$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$, $h \in \mathcal{C}^\alpha(\mathbb{R})$ and for all $x, y \in \overline{\mathbb{R}}$,*

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq (c \|h\|_\alpha + C \|h\|_\infty) d(x, y)^\alpha.$$

Proof. Suppose that the separating condition holds with bounded open interval O . We may assume that $J \subset O$ and that there exists $r_0 > 0$ such that for all $i, j \in I$ with $i \neq j$,

$$\overline{f_i^{-1}(B(O, r_0))} \subset B(O, r_0) \quad \text{and} \quad f_j \left(\overline{f_i^{-1}(B(O, r_0))} \right) \subset \overline{\mathbb{R}} \setminus J,$$

where $B(O, r_0) := \bigcup_{u \in O} B(u, r_0)$. For $i \neq j$ we define the compact sets

$$K_{i,j} := f_j \left(\overline{f_i^{-1}(B(O, r_0))} \right) \quad \text{and} \quad K' := \bigcup_{i,j \in I, i \neq j} K_{i,j} \subset \overline{\mathbb{R}} \setminus J.$$

Let $\delta := d(\overline{\mathbb{R}} \setminus O, J)$. Since $J \subset O$ is compact, we have $\delta > 0$. Define the compact set

$$K := K' \cup \{u \in \overline{\mathbb{R}} \setminus J, d(u, J) \geq \delta/2\} \subset \overline{\mathbb{R}} \setminus J.$$

Since \overline{O} is compact, by modifying r_0 if necessary, we may assume that

$$(2.4) \quad \forall x' \in \overline{O} \forall y' \in B(x', r_0) \forall i \in I: d(f_i(x'), f_i(y')) < \delta/2.$$

Let $x, y \in \overline{\mathbb{R}}$ and $n \in \mathbb{N}$ sufficiently large (to be determined later). We now distinguish two cases. First suppose that $x \notin O$. We may assume that $d(x, y) \leq \delta/2$. Hence, x and y are contained in the compact set

$\{u \in \overline{\mathbb{R}} \mid d(u, J) \geq \delta/2\} \subset K \subset \overline{\mathbb{R}} \setminus J$ and belong to the same connected component of $\overline{\mathbb{R}} \setminus J$. Therefore, by Lemma 2.11, we have

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq \sum_{|\tau|=n} p_{\tau} |h(f_{\tau}(x)) - h(f_{\tau}(y))| \leq \sum_{|\tau|=n} p_{\tau} \|h\|_{\alpha} (C_K \eta^n)^{\alpha} d(x, y)^{\alpha} \leq \|h\|_{\alpha} (C_K \eta^n)^{\alpha} d(x, y)^{\alpha}.$$

For n sufficiently large, we have $(C_K \eta^n)^{\alpha} < 1$. This finishes the proof in the case when $x \notin O$.

Next we consider the remaining case $x \in O$. Let

$$\ell_x := \sup \left\{ \ell \geq 1 \mid \exists \omega \in I^* \text{ such that } f_{\omega_j}(x) \in O \text{ for any } 1 \leq j \leq \ell \right\} \in \mathbb{N} \cup \{\infty\}.$$

Then there exists a unique $(\omega_1, \dots, \omega_{\ell_x}) \in I^{\ell_x}$ such that $f_{\omega_{\ell_x}} \cdots f_{\omega_1}(x) \in O$. We distinguish two cases (a) $\ell_x \geq n$ and (b) $\ell_x < n$. We begin with case (a). Define

$$B_n(x) := f_{\omega_n}^{-1}(B(f_{\omega_n}(x), r_0)).$$

We may assume that $y \in B_n(x)$. We have the decomposition

$$I^n = \bigcup_{j \leq n} \bigcup_{|\tau|=n-j, \tau_1 \neq \omega_{j+1}} \{\omega_j \tau\}.$$

For all $j < n$ and $\tau_1 \in I \setminus \{\omega_{j+1}\}$ we have

$$f_{\tau_1}(f_{\omega_j}(B_n(x))) \subset f_{\tau_1}(f_{\omega_{j+1}}^{-1}(O, r_0)) \subset K' \subset \overline{\mathbb{R}} \setminus J.$$

By Lemma 2.11 we have

$$\begin{aligned} d\left(f_{\tau_1}(f_{\omega_j}(x)), f_{\tau_1}(f_{\omega_j}(y))\right) &\leq C_{K'} \eta^{n-j-1} d\left(f_{\tau_1 \omega_j}(x), f_{\tau_1 \omega_j}(y)\right) \\ &\leq C_{K'} \text{Lip}(f_{\tau_1}) \eta^{n-j-1} d\left(f_{\omega_j}(x), f_{\omega_j}(y)\right). \end{aligned}$$

Put $C_1 := C_{K'} \max_{i \in I} \text{Lip}(f_i)$. Since h is Hölder continuous, we can assert that

$$\begin{aligned} |M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| &= \left| \sum_{j \leq n} p_{\omega_j} \sum_{|\tau|=n-j, \tau_1 \neq \omega_{j+1}} p_{\tau} \left(h\left(f_{\tau}(f_{\omega_j}(x))\right) - h\left(f_{\tau}(f_{\omega_j}(y))\right) \right) \right| \\ &\leq \sum_{j \leq n} p_{\omega_j} \sum_{|\tau|=n-j, \tau_1 \neq \omega_{j+1}} p_{\tau} \|h\|_{\alpha} C_1^{\alpha} (\eta^{n-j-1})^{\alpha} d\left(f_{\omega_j}(x), f_{\omega_j}(y)\right)^{\alpha} \\ &= C_1^{\alpha} \|h\|_{\alpha} \sum_{j \leq n} p_{\omega_j} (\eta^{n-j-1})^{\alpha} d\left(f_{\omega_j}(x), f_{\omega_j}(y)\right)^{\alpha}. \end{aligned}$$

Using that d is strongly equivalent to the Euclidean metric on the compact set $\overline{B(O, r_0)} \subset \mathbb{R}$ and combining with the bounded distortion estimate in (2.3) of Lemma 2.13 with $\Omega := B(O, r_0)$, we deduce the existence of $D_0 < \infty$ such that

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq C_1^{\alpha} D_0^{\alpha} \|h\|_{\alpha} \sum_{j \leq n} p_{\omega_j} (\eta^{n-j-1})^{\alpha} |f'_{\omega_j}(x)|^{\alpha} d(x, y)^{\alpha}.$$

The following was proved in [JS17, (6.2) in the proof of Theorem 1.3]. There exists a $C(\varphi, \psi_{\mathbf{p}})$ such that for all $j \in \mathbb{N}$ and for all $\omega \in \Sigma$,

$$(2.5) \quad e^{S_j \psi_{\mathbf{p}}(\omega)} \leq C(\varphi, \psi_{\mathbf{p}}) e^{\alpha_{-}(\mathbf{p}) S_j \varphi(\omega)}.$$

It is easy to see that $C(\varphi, \psi_{\mathbf{p}})$ depends continuously on $\mathbf{p} \in (0, 1)^s$. Let $\omega = (\omega_1, \dots, \omega_{\ell_x}, \omega_1, \dots, \omega_{\ell_x}, \dots) \in \Sigma$. By combining (2.5) with the bounded distortion estimate in (2.3) we have for some D_1 with $D_1 > D_0$

$$p_{\omega_j} \leq D_1^{-\alpha_{-}(\mathbf{p})} C(\varphi, \psi_{\mathbf{p}}) |f'_{\omega_j}(x)|^{-\alpha_{-}(\mathbf{p})}.$$

Hence, we obtain

$$|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| \leq C_1^\alpha D_1^{\alpha + \alpha_-(\mathbf{p})} C(\varphi, \psi_{\mathbf{p}}) \|h\|_\alpha \sum_{j \leq n} \eta^{\alpha(n-j-1)} |f'_{\omega_j}(x)|^{\alpha - \alpha_-(\mathbf{p})} d(x, y)^\alpha.$$

For $\delta > 0$ sufficiently small and $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ we have $\sup_{\mathbf{p} \in B(\mathbf{p}_0, \delta)} \alpha - \alpha_-(\mathbf{p}) < 0$ by Lemma 2.10. Also we can define $C(\varphi, \psi) := \sup_{\mathbf{p} \in B(\mathbf{p}_0, \delta)} C(\varphi, \psi_{\mathbf{p}}) < \infty$. Since $|f'_{\omega_j}(x)| \geq \lambda^j$ there exist $\tilde{\eta} < 1$ and $C_2 < \infty$ such that for all $j \leq n$ and $\mathbf{p} \in B(\mathbf{p}_0, \delta)$,

$$\eta^{\alpha(n-j-1)} |f'_{\omega_j}(x)|^{\alpha - \alpha_-(\mathbf{p})} \leq C_2 \tilde{\eta}^n.$$

Therefore,

$$|M^n h(x) - M^n h(y)| \leq C_1^\alpha D_1^{\alpha + \alpha_-(\mathbf{p})} C(\varphi, \psi) \|h\|_\alpha C_2 n \tilde{\eta}^n d(x, y)^\alpha.$$

Put $c := C_1^\alpha D_1^{\alpha + \alpha_-(\mathbf{p})} C(\varphi, \psi) \|h\|_\alpha C_2 n \tilde{\eta}^n$. For n sufficiently large we have $c < 1$. Thus, assuming case (a), we have derived the desired estimate.

Finally, to complete the proof, let us consider the case (b) when $\ell_x < n$. We may assume that $y \in B_{\ell_x}(x)$. We estimate

$$\begin{aligned} |M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)| &\leq \left| \sum_{j < \ell_x} p_{\omega_j} \sum_{|\tau|=n-j, \tau_1 \neq \omega_{j+1}} p_\tau \left(h \left(f_\tau \left(f_{\omega_j}(x) \right) \right) - h \left(f_\tau \left(f_{\omega_j}(y) \right) \right) \right) \right| \\ &\quad + p_{\omega_{\ell_x}} \left| \sum_{|\tau|=n-\ell_x} p_\tau \left(h \left(f_\tau \left(f_{\omega_{\ell_x}}(x) \right) \right) - h \left(f_\tau \left(f_{\omega_{\ell_x}}(y) \right) \right) \right) \right|. \end{aligned}$$

The first summand on the right-hand side satisfies an α -Hölder condition with $c < 1$ for n large by the same arguments as in (a) above. Finally, to deal with the second summand, let $x' := f_{\omega_{\ell_x}}(x)$ and $y' := f_{\omega_{\ell_x}}(y)$. Since $y \in B_{\ell_x}(x)$ we have that $d(x', y') < r_0$. By the definition of ℓ_x we have that $f_{\tau_1}(x') \notin O$ for all $\tau_1 \in I$. By (2.4) we have $d(f_{\tau_1}(y'), J) \geq \delta/2 > 0$. Since moreover $f_{\tau_1}(x')$ and $f_{\tau_1}(y')$ are in the same component of $\mathbb{R} \setminus J$, Lemma 2.11 implies that

$$d \left(f_\tau \left(f_{\omega_{\ell_x}}(x) \right), f_\tau \left(f_{\omega_{\ell_x}}(y) \right) \right) \leq C_K \eta^{n-\ell_x} d(f_{\omega_{\ell_x}}(x), f_{\omega_{\ell_x}}(y)).$$

The rest of the proof runs as in (a) above. The proof is complete. \square

We thus obtain the following strengthening of Theorem 2.4 when the separating condition holds.

Theorem 2.15. *Suppose that $(f_i)_{i \in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0, 1)^s$ and $\alpha < \alpha_-(\mathbf{p}_0)$. There exists $\delta > 0$ such that $M_{\mathbf{p}} : \mathcal{C}^\alpha(\mathbb{R}) \rightarrow \mathcal{C}^\alpha(\mathbb{R})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. Moreover, the map $\mathbf{p} \mapsto T_{\mathbf{p}} \in \mathcal{C}^\alpha(\mathbb{R})$ is real-analytic on $B(\mathbf{p}_0, \delta)$.*

Corollary 2.16. *Suppose that $(f_i)_{i \in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0, 1)^s$ and $\alpha < \alpha_-(\mathbf{p}_0)$. There exists $\delta > 0$ such that $\mathcal{T}_{\mathbf{p}} \subset \mathcal{C}^\alpha(\mathbb{R})$ for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$.*

Remark 2.17. By Proposition 4.5 below it will turn out that in many cases, $\mathcal{T}_{\mathbf{p}} \not\subset \mathcal{C}^{\alpha_-(\mathbf{p})}(\mathbb{R})$. Hence, by the previous corollary, the spectral gap property stated in Theorem 2.15 is sharp. In particular, this is the case for the classical Takagi function. Namely, let $f_1(x) = 2x$, $f_2(x) = 2x - 1$. Then $J = [0, 1]$ and $T_{1/2}(x) = x$ for $x \in J$ and C_1 is a multiple of the classical Takagi function on J . Hence, $\alpha_- = 1$, $T_{1/2}$ is Lipschitz continuous, and C_1 is α -Hölder continuous for every $\alpha < 1$, but not Lipschitz continuous. For further examples, see Proposition 4.5.

3. MULTIFRACTAL ANALYSIS OF THE POINTWISE HÖLDER EXPONENT

In this section we perform a complete multifractal analysis of the pointwise Hölder exponents of the elements of \mathcal{T} associated with $(f_i)_{i \in I}$ and $\mathbf{p} \in (0, 1)^s$. We begin by providing the necessary terminology which has been introduced in [JS17]. We use $\mathbf{n} = (n_1, \dots, n_s)$ to denote an element of \mathbb{N}_0^s . Let $\mathbf{e}_i \in \mathbb{N}_0^s$ denote the element whose i th component is 1 and all other components are 0. An element $A \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ is represented as $A = (A_{\mathbf{x}, \mathbf{y}})_{(\mathbf{x}, \mathbf{y}) \in \mathbb{N}_0^s \times \mathbb{N}_0^s}$, where $A_{\mathbf{x}, \mathbf{y}} \in \mathbb{R}$, and such an element A is called an (\mathbb{N}_0^s) -matrix. $A_{\mathbf{x}, \mathbf{y}}$ is called the (\mathbf{x}, \mathbf{y}) -component of A . We denote by $\mathbf{1}_{\mathbf{n}, \mathbf{m}} \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ the matrix such that for every $(\mathbf{x}, \mathbf{y}) \in \mathbb{N}_0^s \times \mathbb{N}_0^s$ the (\mathbf{x}, \mathbf{y}) -component of $\mathbf{1}_{\mathbf{n}, \mathbf{m}}$ is given by

$$(\mathbf{1}_{\mathbf{n}, \mathbf{m}})_{\mathbf{x}, \mathbf{y}} = \begin{cases} 1, & \mathbf{n} = \mathbf{x}, \mathbf{m} = \mathbf{y} \\ 0, & \text{else.} \end{cases}$$

In order to investigate \mathcal{T} we define the matrix cocycle $A_0 : \Sigma \times \mathbb{N} \rightarrow \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ given by

$$A_0(\omega, 1) := \begin{cases} \sum_{\mathbf{n} \in \mathbb{N}_0^s} (p_{\omega_1} \mathbf{1}_{\mathbf{n}, \mathbf{n}} + n_{\omega_1} \mathbf{1}_{\mathbf{n}, \mathbf{n} - \mathbf{e}_{\omega_1}}), & \omega_1 \in \{1, \dots, s\} \\ \sum_{\mathbf{n} \in \mathbb{N}_0^s} (p_{\omega_1} \mathbf{1}_{\mathbf{n}, \mathbf{n}} - \sum_{i=1}^s n_i \mathbf{1}_{\mathbf{n}, \mathbf{n} - \mathbf{e}_i}), & \omega_1 = s+1 \end{cases}$$

and for $k \in \mathbb{N}$,

$$A_0(\omega, k) := A_0(\omega, 1)A_0(\sigma\omega, 1) \dots A_0(\sigma^{k-1}\omega, 1) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

Here, the matrix product $A_0(\tau, 1)A_0(\nu, 1) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ is for $\tau, \nu \in I^{\mathbb{N}}$ and $\mathbf{l}, \mathbf{m} \in \mathbb{N}_0^s$ given by

$$(3.1) \quad (A_0(\tau, 1) \cdot A_0(\nu, 1))_{\mathbf{l}, \mathbf{m}} := \sum_{\mathbf{k} \in \mathbb{N}_0^s} (A_0(\tau, 1))_{\mathbf{l}, \mathbf{k}} \cdot (A_0(\nu, 1))_{\mathbf{k}, \mathbf{m}}.$$

Note that the sum in (3.1) is actually a finite sum. Further matrix products in the definition of $A_0(\omega, k)$ are defined in the same way. We also define

$$A(\omega, k) := (p_{\omega_k})^{-1} A_0(\omega, k) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

Moreover, for all $a, b \in \mathbb{R}$ we define the matrix

$$U(a, b) := (u_{\mathbf{n}}(a, b))_{\mathbf{n} \in \mathbb{N}_0^s} \in \mathbb{R}^{\mathbb{N}_0^s} \quad \text{given by } u_{\mathbf{n}}(a, b) := C_{\mathbf{n}}(a) - C_{\mathbf{n}}(b).$$

Remark. In (3.1) and in the following we make use of the product of matrices with an infinite index set. These matrix products will always be well defined, since either the first factor of the product possesses at most finitely many non-zero entries in each row, or the second factor contains at most finitely many non-zero entries in each column.

Definition 3.1. We say that $(f_i)_{i \in I}$ satisfies the open set condition if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$ we have $f_i^{-1}(O) \cap f_j^{-1}(O) = \emptyset$.

We write $A \leq B$ for subsets $A, B \subset \mathbb{R}$ if $a \leq b$ for every $a \in A$ and $b \in B$.

Remark. If $(f_i)_{i \in I}$ satisfies the open set condition, then we will always assume that $f_i^{-1}(\overline{O}) \leq f_j^{-1}(\overline{O})$ for all $i, j \in I$ with $i < j$.

The purpose of the above definitions is the following.

Lemma 3.2. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $k \in \mathbb{N}$, $\omega \in I^k$ and $x, y \in f_{\omega}^{-1}(\overline{O})$. Then we have $U(x, y) = A_0(\overline{\omega}, k)U(f_{\omega}(x), f_{\omega}(y))$.*

Proof. The assertion can be shown as in [JS17, Lemma 4.7] if we observe that $u_{\mathbf{n}}(f_{\tau}(x), f_{\tau}(y)) = 0$, for all $\tau \in I^k$ with $\tau \neq \omega$ and for all $\mathbf{n} \in \mathbb{N}_0^s$. To prove this, note that by the open set condition we have that $[f_{\tau}(x), f_{\tau}(y)] \cap \overline{O}$ has at most one point. Consequently, $T_{\mathbf{p}}$ is constant on $[f_{\tau}(x), f_{\tau}(y)]$ and thus every $C_{\mathbf{n}}$ is constant on $[f_{\tau}(x), f_{\tau}(y)]$, $\mathbf{n} \in \mathbb{N}_0^s$. \square

Recall that $G = \langle f_1, \dots, f_{s+1} \rangle$ and write $G(x) := \{g(x) \mid g \in G\}$. Using Fact 2.9 it is straightforward to prove the following.

Lemma 3.3. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $x_0 \in J$. Then there exists $a \in (J \cap O) \setminus G(x_0)$ and $b \in O \setminus G(x_0)$ such that b is arbitrarily close to a and $T_{\mathbf{p}}(a) \neq T_{\mathbf{p}}(b)$.*

Remark 3.4. By using Lemma 3.3 we can extend the methods used in [JS17, Lemma 4.8, 4.9, 4.10, Proposition 4.11, Lemma 5.2] to our setting.

The following lemma can be proved as in [JS17, Lemma 4.8]. For $\mathbf{q} = (q_1, \dots, q_s) \in \mathbb{N}_0^s$ we set $|\mathbf{q}| := \sum_{i=1}^s q_i$.

Lemma 3.5. *There exists a constant $K \geq 1$ which depends only on $\mathbf{p} \in (0, 1)^s$ and $\mathbf{q} \in \mathbb{N}^s$ such that for all $\tau \in \Sigma$ and all $k \in \mathbb{N}$,*

$$|A(\tau, k)_{\mathbf{q}, \mathbf{r}}| \leq Kk^{|\mathbf{q}|}.$$

Moreover, we can derive the following key lemma from the proof of [JS17, Lemma 5.2], Lemma 3.3 and Remark 3.4. Note that in [JS17, Lemma 5.2] the Julia set J_{ω} should be replaced by $J(G)$. An element $C \in \mathcal{T}$ is called non-trivial if there exists $(\beta_{\mathbf{n}})_{\mathbf{n}} \neq 0$ such that $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}}$.

Lemma 3.6. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathcal{T}$ be non-trivial. Let $j(k) \rightarrow \infty$ be a sequence of positive integers. Let $\omega \in \Sigma$. In any non-empty neighbourhood V in \mathbb{R} which intersects J there exist $a, b \in V \cap O$ with $a \neq b$ such that*

$$\eta := \limsup_{k \rightarrow \infty} \left| \sum_{\mathbf{m}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} A(\omega, j(k))_{\mathbf{n}, \mathbf{m}} u_{\mathbf{m}}(a, b) \right| \in (0, \infty].$$

3.1. Dynamical characterisation of pointwise Hölder exponents. In this section we provide a dynamical characterisation of the pointwise Hölder exponent of non-trivial elements of \mathcal{T} . The *pointwise Hölder exponent* of C at $x \in \mathbb{R}$ is denoted by $\text{Höl}(C, x)$. An elementary calculation (see [JS15, Lemma 5.1]) shows that for every $x \in \mathbb{R}$,

$$(3.2) \quad \text{Höl}(C, x) = \liminf_{r \rightarrow 0} \frac{\log \sup_{y \in B(x, r)} |C(y) - C(x)|}{\log r}.$$

We start with the upper bound.

Proposition 3.7. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathcal{T}$ be non-trivial. For every $x \in J$ we have*

$$\text{Höl}(C, x) \leq \inf_{\omega \in \pi^{-1}(x)} \liminf_{n \rightarrow \infty} \frac{S_n \Psi(\omega)}{S_n \Phi(\omega)}.$$

Proof. Let $x \in J$ and $\omega \in \pi^{-1}(x)$. Since J is compact, there exists a sequence (j_k) tending to infinity and $x_0 \in J$ such that

$$\alpha := \liminf_{n \rightarrow \infty} \frac{S_n \Psi(\omega)}{S_n \Phi(\omega)} = \lim_{k \rightarrow \infty} \frac{S_{j_k} \Psi(\omega)}{S_{j_k} \Phi(\omega)} \quad \text{and} \quad \lim_{k \rightarrow \infty} f_{\omega_{j_k}}(x) = x_0 \in J.$$

By Lemma 3.6 there exists $\varepsilon > 0$, $\eta > \eta_0 > 0$ and points $a, b \in B(x_0, \varepsilon) \cap O$ with $a \neq b$ such that for all k sufficiently large,

$$\left| \sum_{\mathbf{m}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} A(\omega, j(k))_{\mathbf{n}, \mathbf{m}} u_{\mathbf{m}}(a, b) \right| \geq \eta_0 > 0.$$

Define $y_k := (f_{\omega_{j(k)}})^{-1}(a)$ and $z_k := (f_{\omega_{j(k)}})^{-1}(b)$. By Lemma 3.2 we have

$$C(y_k) - C(z_k) = \sum_{\mathbf{n}} \beta_{\mathbf{n}} (U(y_k, z_k))_{\mathbf{n}} = p_{\omega_{j(k)}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} (A(\omega, j(k))U(a, b))_{\mathbf{n}}.$$

Hence, we can estimate

$$\liminf_{k \rightarrow \infty} \frac{\log |C(y_k) - C(z_k)|}{\log |y_k - z_k|} \leq \liminf_{k \rightarrow \infty} \frac{S_{j(k)} \psi(\omega) + \log \eta_0}{S_{j(k)} \varphi(\omega)} = \alpha.$$

Finally, proceeding exactly as in [JS17, Proof of Lemma 5.3 (5.4),(5.5) and (5.6)], the statement of the lemma follows. \square

Since $\text{Höl}(C, x) < \infty$ by Proposition 3.7, we can conclude the following.

Corollary 3.8. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathcal{T}$ be non-trivial. Then C is not locally constant at any point of J and thus \mathcal{T} is the direct sum of vector spaces $\bigoplus_{\mathbf{n} \in \mathbb{N}_0^s} \mathbb{R} C_{\mathbf{n}}$.*

Next we provide lower bounds for the pointwise Hölder exponent. In the following proposition we define for $\omega \in \Sigma$ and $x = \pi(\omega)$ the sequences

$$\delta_n := \delta_n(\omega) := d(f_{\omega_n}(x), \partial O), \quad \text{and } s_n := s_n(\omega) := \delta_n \cdot \left| f'_{\omega_n}(x) \right|^{-1} \cdot D^{-1},$$

where ∂O refers to the boundary of the open interval O and $D = D(O)$ refers to the bounded distortion constant in Lemma 2.13.

Proposition 3.9. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C \in \mathcal{T} \setminus \{0\}$. Let $\omega \in \Sigma$ be a sequence which is not eventually constant and let $x = \pi(\omega) \in J$. Then we have*

$$\text{Höl}(C, x) \cdot \left(1 + \frac{\liminf_{n \rightarrow \infty} n^{-1} \log \delta_n}{\max \varphi} \right) \geq \liminf_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)}.$$

Proof. By (3.2) there exists a sequence (r_k) tending to zero such that

$$\text{Höl}(C, x) = \lim_{k \rightarrow \infty} \frac{\log \sup_{y \in B(x, r_k)} |C(x) - C(y)|}{\log r_k}.$$

Let $n_k := \max \{n \geq 1 \mid s_n > r_k\}$. Since ω is not eventually constant, we have $s_n > 0$. Hence, $(n_k) \rightarrow \infty$, as $k \rightarrow \infty$. By the definition of n_k we have

$$B(x, r_k) \subset B_k := \left(f_{\omega_{n_k}} \right)^{-1} \left(B \left(f_{\omega_{n_k}}(x), \delta_{n_k} \right) \right), \quad k \in \mathbb{N}.$$

Hence, if $r_k < 1$ then

$$\frac{\log \sup_{y \in B(x, r_k)} |C(x) - C(y)|}{\log r_k} \geq \frac{\log \sup_{y \in B_k} |C(x) - C(y)|}{\log r_k}.$$

Also, by the definition of n_k , we have $s_{n_k+1} \leq r_k$. Therefore, we have

$$\frac{\log \sup_{y \in B_k} |C(x) - C(y)|}{\log r_k} \geq \frac{\log \sup_{y \in B_k} |C(x) - C(y)|}{\log s_{n_k+1}} = \frac{\log \sup_{y \in B_k} |C(x) - C(y)|}{-\log D + S_{n_k+1} \varphi(\omega) + \log \delta_{n_k+1}}.$$

By Lemma 3.2 and Lemma 3.5 there exist constants K' and q , which are independent of k , such that

$$\sup_{y \in B_k} |C(x) - C(y)| \leq K' p_{\omega_{n_k}} n_k^q.$$

We conclude that

$$\begin{aligned} \frac{\log \sup_{y \in B_k} |C(x) - C(y)|}{-\log D + S_{n_k+1} \varphi(\omega) + \log \delta_{n_k+1}} &\geq \frac{S_{n_k} \psi(\omega) + q \log(n_k) + \log K'}{-\log D + S_{n_k+1} \varphi(\omega) + \log \delta_{n_k+1}} \\ &= \frac{S_{n_k} \psi(\omega) \cdot (1 + (q \log(n_k) + \log K') / S_{n_k} \psi(\omega))}{S_{n_k+1} \varphi(\omega) \cdot (1 + (-\log D + \log \delta_{n_k+1}) / S_{n_k+1} \varphi(\omega))} \end{aligned}$$

and the claim follows by letting k tend to infinity. \square

Corollary 3.10. *Suppose that $(f_i)_{i \in I}$ satisfies the separating condition. Let $C \in \mathcal{T} \setminus \{0\}$. Then for all $\omega \in \Sigma$ and $x = \pi(\omega)$, we have*

$$\liminf_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} = \text{Höl}(C, x).$$

Proof. Since $\inf_{n \in \mathbb{N}} \delta_n > 0$, we have $\liminf_{n \rightarrow \infty} n^{-1} \log \delta_n = 0$. Hence, $\liminf_{n \rightarrow \infty} S_n \psi(\omega) / S_n \varphi(\omega) \leq \text{Höl}(C, x)$ by Proposition 3.9. The converse inequality follows from Proposition 3.7. \square

Let us end this section with the following remark concerning Proposition 3.9 and Corollary 3.10.

Remark 3.11. It is not difficult to find examples of systems $(f_i)_{i \in I}$ satisfying the open set condition with limit points $x = \pi(\omega) \in J$, $\omega \in \Sigma$, such that

$$\text{Höl}(C_0, x) < \lim_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)}.$$

Hence, in contrast to systems satisfying the separating condition (see Corollary 3.10), the dynamical characterization of the pointwise Hölder exponent in terms of quotients of ergodic sums does not always hold for systems satisfying the open set condition. However, Proposition 3.9 can be used to establish a dynamical characterization for almost every limit point with respect to suitable reference measures. Moreover, it turns out in Theorem 3.14 below that the dimension spectrum of the pointwise Hölder exponents coincides with the spectrum of quotients of ergodic sums of the potentials φ and ψ .

3.2. Dimension spectrum of pointwise Hölder exponents. We define

$$\mathcal{F}(\alpha) := \mathcal{F}_{\mathbf{p}}(\alpha) := \pi \left\{ \omega \in \Sigma \mid \lim_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} = \alpha \right\}.$$

Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. It is well known that the multifractal spectrum is complete ([Sch99]), that is, there exist $\alpha_-, \alpha_+ \in \mathbb{R}$ such that $\mathcal{F}(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_-, \alpha_+]$. In fact, we have

$$\alpha_- = \inf_{\omega \in \Sigma} \liminf_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} \quad \alpha_+ = \inf_{\omega \in \Sigma} \limsup_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)}.$$

For every $\beta \in \mathbb{R}$ there exists a unique $t(\beta) \in \mathbb{R}$ such that $\mathcal{P}(t(\beta)\varphi + \beta\psi) = 0$, where $\mathcal{P}(u)$ refers to the topological pressure of a continuous function u with respect to the dynamical system (Σ, σ) (see [Wal82]). It is well-known that the function t is real-analytic and convex function with $t'(\beta) = -\int \psi d\mu_\beta / \int \varphi d\mu_\beta$ where μ_β denotes the unique Gibbs probability measure on Σ associated with $t(\beta)\varphi + \beta\psi$. The function t is strictly convex if and only if $\alpha_- < \alpha_+$, and have that $\alpha_- = \alpha_+$ if and only if $\delta\varphi$ and ψ are cohomologous, where

$$\delta := t(0) = \dim_H(J).$$

Here, we say that $\delta\varphi$ and ψ are cohomologous if there exists a continuous function $\kappa : \Sigma \rightarrow \mathbb{R}$ such that $\delta\varphi = \psi + \kappa - \kappa \circ \sigma$. Note that we have $-t'(\mathbb{R}) = (\alpha_-, \alpha_+)$ if $\alpha_- < \alpha_+$, and $-t'(\mathbb{R}) = \{\alpha_-\}$, otherwise.

We also define the level sets

$$\mathcal{F}^\#(\alpha) := \begin{cases} \pi \left\{ \omega \in \Sigma \mid \limsup_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} \geq \alpha \right\}, & \alpha \geq \alpha_0 \\ \pi \left\{ \omega \in \Sigma \mid \liminf_{n \rightarrow \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} \leq \alpha \right\}, & \alpha < \alpha_0, \end{cases}$$

where we have set $\alpha_0 := \int \psi d\mu_0 / \int \varphi d\mu_0$. We denote the convex conjugate of t ([Roc70]) by

$$t^*(u) := \sup \{ \beta u - t(\beta) \mid \beta \in \mathbb{R} \} \in \mathbb{R} \cup \{+\infty\}.$$

It is well-known (see e.g. [Pes97, Sch99]) that for $\alpha \in [\alpha_-, \alpha_+]$,

$$(3.3) \quad \dim_H(\mathcal{F}(\alpha)) = \dim_H(\mathcal{F}^\#(\alpha)) = -t^*(-\alpha) \geq 0,$$

and that $\mathcal{F}(\alpha) = \mathcal{F}^\#(\alpha) = \emptyset$ for $\alpha \notin [\alpha_-, \alpha_+]$. To prove this, it is shown that if $\beta \in \mathbb{R}$ and $\alpha = -t'(\beta)$, then for the corresponding Gibbs measure μ_β we have $\mu_\beta \circ \pi^{-1}(\mathcal{F}(\alpha)) = 1$ and

$$(3.4) \quad \dim_H(\mathcal{F}(\alpha)) = \dim_H(\mu_\beta \circ \pi^{-1}) > 0.$$

We refer to [JS15, JS17] for a closely related framework for random complex dynamical systems. See in particular [JS15, Rem. 3.14, Prop. 4.4, Thm. 5.13]. If $\alpha_- = \alpha_+$ then $\mathcal{F}(\alpha_-) = J$ and for every $\beta \in \mathbb{R}$,

$$\dim_H(\mathcal{F}(\alpha_-)) = \dim_H(\mu_\beta \circ \pi_1^{-1}) = \dim_H(J) = t(0) = \delta.$$

Corollary 3.12. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}}$ be non-trivial. Then for all $\alpha \in [\alpha_-, \alpha_+]$ we have*

$$\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} \geq -t^*(-\alpha).$$

Proof. It is well known that, for each $\alpha \in [\alpha_-, \alpha_+]$, there exists an ergodic Borel probability measure μ with $\int \psi d\mu / \int \varphi d\mu = \alpha$ and $\dim_H(\mu \circ \pi^{-1}) = -t^*(-\alpha) \geq 0$ (see e.g. [Pes97, Sch99]). Suppose that $-t^*(-\alpha) > 0$; otherwise there is nothing to prove. Following [Pat97] we have for μ -a.e. $\omega \in \Sigma$ and $x = \pi(\omega)$, $\lim_{n \rightarrow \infty} n^{-1} \log \delta_n = 0$, where δ_n is defined prior to Proposition 3.9. Hence, for μ -a.e. $\omega \in \Sigma$ and $x = \pi(\omega)$, $\text{Höl}(C, x) \geq \alpha$ by Proposition 3.9. Moreover, by Proposition 3.7 we have $\text{Höl}(C, x) \leq \alpha$ μ -a.e. We conclude that $\mu \circ \pi^{-1}(\{x \in J \mid \text{Höl}(C, x) = \alpha\}) = 1$. Thus, $\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} \geq \dim_H(\mu \circ \pi^{-1}) = -t^*(-\alpha)$. \square

The following proposition is an extension of results of Allaart ([All17]) for selfsimilar measures.

Proposition 3.13. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition with open set O . For every $\alpha \in [\alpha_-, \alpha_0]$ and every $C \in \mathcal{T}$ we have*

$$\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} \leq -t^*(-\alpha).$$

Further, if $\alpha < \alpha_-$ then $\{x \in J \mid \text{Höl}(C, x) = \alpha\} = \emptyset$.

Proof. Let $x = \pi(\omega) \in J$ with $\text{Höl}(C, x) = \alpha$. By (3.2) there exists $r_k \rightarrow 0$ such that

$$\alpha = \lim_{k \rightarrow \infty} \log \sup_{y \in B(x, r_k)} |C(y) - C(x)| / \log r_k = \alpha.$$

Define $n_k := \min \{n \geq 1 \mid \text{diam} \pi([\omega_n]) < r_k\}$, $k \geq 1$, where $\text{diam}(A) := \sup \{d(x, y) \mid x, y \in A\}$ refers to the diameter of a set $A \subset \mathbb{R}$. If

$$(\omega_1, \dots, \omega_{n_k}) = (\tau j(s+1)^\ell),$$

for some $\tau \in I^*$, $j \leq s$ and $l \geq 1$ then we define $v_k, v'_k \in I^*$ given by

$$v_k := (\tau j(s+1)^{\ell-1}) = (\omega_1, \dots, \omega_{n_k-1}), \quad v'_k := (\tau(j+1)1^\ell),$$

where $\ell' \geq 1$ is given by

$$\ell' := \ell \left\lfloor \frac{\varphi(\overline{s+1})}{\varphi(\overline{1})} \right\rfloor.$$

Here, for $a \in I$ and $m \in \mathbb{N}$ we have set $a^m := (a, \dots, a) \in I^m$, and for $\gamma \in I^*$, $\overline{\gamma} := (\gamma\gamma\dots) \in I^{\mathbb{N}}$. Similarly, if $(\omega_1, \dots, \omega_{n_k}) = (\tau(j+1)1^\ell)$, then we define $v_k := (\omega_1, \dots, \omega_{n_k-1})$ and $v'_k := (\tau j(s+1)^\ell)$. If $\omega_{n_k} \notin \{1, s+1\}$ then we define

$$v_k := v'_k := (\omega_1, \dots, \omega_{n_k-1}).$$

Let $n'_k := |v'_k|$. Then we have as $k \rightarrow \infty$,

$$\exp(S_{n_k} \varphi(\overline{v_k})) \asymp \exp\left(S_{n'_k} \varphi\left(\overline{v'_k}\right)\right),$$

where, for sequences of positive numbers (a_k) and (b_k) the notation $a_k \asymp b_k$ means that a_k/b_k is bounded away from zero and infinity. It is clear by the definition of n_k , v_k and v'_k that there exists a constant $c > 0$ independent of $x \in J$ and $k \geq 1$ such that

$$B(x, r_k/c) \cap J \subset \pi([v_k]) \cup \pi([v'_k]).$$

Let $\varepsilon > 0$. We will derive from our assumption $\text{Höl}(C, x) = \alpha$ that there exists $N \geq 1$ such that for all $k \geq N$,

$$(3.5) \quad \frac{S_{|v_k|} \psi(\overline{v_k})}{S_{|v_k|} \varphi(\overline{v_k})} \leq \alpha + \varepsilon \quad \text{or} \quad \frac{S_{|v'_k|} \psi(\overline{v'_k})}{S_{|v'_k|} \varphi(\overline{v'_k})} \leq \alpha + \varepsilon.$$

To prove (3.5), we first note that by Lemma 3.2 and 3.5, we have for every $v \in I^*$,

$$\sup_{y_1, y_2 \in \pi([v])} |C(y_1) - C(y_2)| \leq K' \exp(S_{|v|} \psi(\overline{v})) |v|^q,$$

where $K' \geq 0$ and $q \geq 0$ are uniform constants. Suppose for a contradiction that (3.5) does not hold. Then, by passing to a subsequence of (n_k) we may assume that for all k and for all $v \in \{v_k, v'_k\}$ we have $S_{|v|} \psi(\overline{v})/S_{|v|} \varphi(\overline{v}) \geq \alpha + \varepsilon$, and so, taking K' large enough, we have

$$\sup_{y_1, y_2 \in \pi([v])} |C(y_1) - C(y_2)| \leq K' \exp(S_{|v|} \psi(\overline{v})) |v|^q \leq K' r_k^{\alpha+\varepsilon} |v|^q.$$

Since $B(x, r_k/c) \subset \pi([v_k]) \cup \pi([v'_k])$, we conclude that

$$\lim_{k \rightarrow \infty} \log \sup_{y \in B(x, r_k)} |C(y) - C(x)| / \log r_k \geq \alpha + \varepsilon.$$

This contradiction proves (3.5).

Let $\beta > 0$, $\eta > 0$ and let $s = t(\beta) + \beta(\alpha + \varepsilon) + \eta$. We define

$$\mathcal{C}_{\alpha+\varepsilon} := \left\{ \tau \in I^* \mid \frac{S_{|\tau|} \psi(\overline{\tau})}{S_{|\tau|} \varphi(\overline{\tau})} \leq \alpha + \varepsilon \right\}.$$

We obtain a covering of $\{\text{Höl}(C, x) = \alpha\}$ by cylinders of sufficiently small diameters as follows. For each $x \in J$ with $\text{Höl}(C, x) = \alpha$ we define the sequence n_k as above and find $v_k \in I^*$ with $x \in \pi([v_k])$ and satisfying (3.5). To verify that the corresponding sum of diameters $\sum_v \text{diam}(\pi([v]))^s$ converges, we proceed as follows. If $v \notin \mathcal{C}_{\alpha+\varepsilon}$ then, by (3.5), we can replace v by $v' \in \mathcal{C}_{\alpha+\varepsilon}$. Note that $\text{diam}(\pi([v])) \asymp \text{diam}(\pi([v']))$ by construction. Since the involved numbers ℓ and ℓ' are comparable, we have that every element of $\mathcal{C}_{\alpha+\varepsilon}$ is taken at most a uniformly bounded number of times. Since

$$(3.6) \quad \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} \text{diam}(\pi([\omega]))^s \asymp \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} e^{(t(\beta)+\beta(\alpha+\varepsilon)+\eta)S_{|\omega|} \varphi(\overline{\omega})} \leq \sum_{\omega \in \mathcal{C}_{\alpha+\varepsilon}} e^{(t(\beta)+\eta)S_{|\omega|} \varphi(\overline{\omega}) + \beta S_{|\omega|} \psi(\overline{\omega})} < \infty,$$

we therefore conclude that the s -dimensional Hausdorff measure of $\{x \in J \mid \text{Höl}(C, x) = \alpha\}$ is finite. Now, first assume that $\alpha \in [\alpha_-, \alpha_0]$. Since ε and η are arbitrary, it follows that

$$\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} \leq \inf_{\beta > 0} \{t(\beta) + \beta\alpha\} = -t^*(-\alpha),$$

where we have used $\alpha \leq \alpha_0$ for the last equality. Finally, if $\alpha < \alpha_-$ then $\mathcal{C}_{\alpha+\varepsilon} = \emptyset$ if $\alpha + \varepsilon < \alpha_-$. By the above construction of the covering of $\{x \in J \mid \text{Höl}(C, x) = \alpha\}$, it thus follows that $\{x \in J \mid \text{Höl}(C, x) = \alpha\} = \emptyset$. The proof is complete. \square

Theorem 3.14. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C \in \mathcal{T} \setminus \{0\}$. Then we have for all $\alpha \in [\alpha_-, \alpha_+]$,*

$$\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} = -t^*(-\alpha),$$

and for $\alpha \notin [\alpha_-, \alpha_+]$ we have $\{x \in J \mid \text{Höl}(C, x) = \alpha\} = \emptyset$.

Proof. By Corollary 3.12, we have $\dim_H \{x \in J \mid \text{Höl}(C, x) = \alpha\} \geq -t^*(-\alpha)$ for $\alpha \in [\alpha_-, \alpha_+]$. By Proposition 3.7 we have for every $\alpha \in \mathbb{R}$,

$$(3.7) \quad \{x \in J \mid \text{Höl}(C, x) = \alpha\} \subset \pi \left(\left\{ \omega \in \Sigma \mid \liminf_{n \rightarrow \infty} \frac{S_n \Psi(\omega)}{S_n \varphi(\omega)} \geq \alpha \right\} \right).$$

We distinguish two cases. If $\alpha \geq \alpha_0$, then by the definition of $\mathcal{F}^\#(\alpha)$ we have $\{x \in J \mid \text{Höl}(C, x) = \alpha\} \subset \mathcal{F}^\#(\alpha)$. Hence, by (3.3), we have $\dim_H (\{x \in J \mid \text{Höl}(C, x) = \alpha\}) \leq \dim_H (\mathcal{F}^\#(\alpha)) = -t^*(-\alpha)$ for $\alpha \geq \alpha_0$. Also, by (3.7), we have $\{x \in J \mid \text{Höl}(C, x) = \alpha\} = \emptyset$ if $\alpha > \alpha_+$. For $\alpha \leq \alpha_0$ the remaining assertions follow from Proposition 3.13. The proof is complete. \square

Remark 3.15. In [BKK16] the pointwise Hölder exponent of affine zipper curves generated by contracting affine mappings is investigated. The multifractal dimension spectrum is obtained only for $\alpha \geq \alpha_0$. In a recent preprint of Allaart ([All17]) the complete multifractal spectrum of pointwise Hölder exponents for curves associated with selfsimilar measures is obtained.

4. GLOBAL HÖLDER CONTINUITY

In this section we investigate the global Hölder continuity of the elements of \mathcal{T} . The first statement of the next theorem has been obtained for the Minkowski's question mark function in [KS08], for distributions of conformal iterated function systems satisfying the separating condition in [KS09], and for expanding circle diffeomorphisms in [JKPS09].

Theorem 4.1. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Then we have the following.*

- (1) $T_{\mathbf{p}} \in \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$.
- (2) $\mathcal{T} \subset \bigcap_{\alpha < \alpha_-} \mathcal{C}^\alpha(\overline{\mathbb{R}})$.

Proof. We only verify the desired Hölder continuity at points $x, y \in J$. That this is sufficient can be seen as follows. If $y \in \overline{\mathbb{R}} \setminus J$, then either there exists $u \in J$ between x and y and the desired Hölder continuity follows from the triangle inequality, or we have $T_{\mathbf{p}}(x) = T_{\mathbf{p}}(y)$ (resp. $C(x) = C(y)$). We may assume that $O = (\text{Fix}(f_1), \dots, \text{Fix}(f_{s+1}))$, where $\text{Fix}(f_k)$ denotes the unique fixed point of f_k in \mathbb{R} .

Let $x, y \in J$ with $x < y$. Let $\omega \in I^* \cup \{\emptyset\}$ and $i, j \in I$ with $i < j$ such that $f_\omega(x) \in \overline{O}$, $f_\omega(y) \in \overline{O}$ and $f_i(f_\omega(x)) \in \overline{O}$ and $f_j(f_\omega(y)) \in \overline{O}$. Note that such ω always exists because $x, y \in J$ and $x < y$.

We first consider the case when $j = i + 1$. Let

$$\ell := \sup \{n \geq 0 \mid f_{s+1}^n(f_i(f_\omega(x))) \in \overline{O}\} \quad \text{and} \quad \ell' := \sup \{n \geq 0 \mid f_1^n(f_j(f_\omega(y))) \in \overline{O}\}.$$

We define

$$\xi := \max(f_{\omega_i})^{-1}(\bar{O}) \quad \text{and} \quad \xi' := \min(f_{\omega_j})^{-1}(\bar{O}).$$

We will verify that there exists a uniform constant $D' \geq 1$ such that

$$(4.1) \quad |T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| \leq C(\varphi, \psi_{\mathbf{p}}) D' d(x, \xi)^{\alpha_{-}(\mathbf{p})}$$

and

$$(4.2) \quad |T_{\mathbf{p}}(\xi') - T_{\mathbf{p}}(y)| \leq C(\varphi, \psi_{\mathbf{p}}) D' d(\xi', y)^{\alpha_{-}(\mathbf{p})}.$$

We can then conclude that

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(y)| \leq |T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| + |T_{\mathbf{p}}(\xi) - T_{\mathbf{p}}(\xi')| + |T_{\mathbf{p}}(\xi') - T_{\mathbf{p}}(y)|,$$

which proves the first assertion in the case when $j = i + 1$ because $T_{\mathbf{p}}$ is constant on $[\xi, \xi']$.

We only verify (4.1), the proof of (4.2) is complete analogous. To prove (4.1) we may assume that $\ell < \infty$ because if $\ell = \infty$ then $x = \xi$ and (4.1) holds trivially. By the definition of ℓ there exists $1 \leq k < s + 1$ such that

$$x \in f_{\omega}^{-1} f_i^{-1} f_{s+1}^{-\ell} f_k^{-1}(\bar{O}) \leq f_{\omega}^{-1} f_i^{-1} f_{s+1}^{-\ell} f_{s+1}^{-1}(\bar{O}) \subset f_{\omega}^{-1} f_i^{-1}(\bar{O}).$$

We conclude that

$$(4.3) \quad d(x, \xi) \geq \text{diam}\left(f_{\omega}^{-1} f_i^{-1} f_{s+1}^{-\ell} f_{s+1}^{-1}(\bar{O})\right).$$

Define $\tau := \omega_i(s+1)^{\ell}$. Clearly, we have $[x, \xi] \subset f_{\tau}^{-1}(\bar{O})$. Since the open set condition holds,

$$(4.4) \quad |T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| \leq p_{\tau}.$$

By (2.5) there exists a constant $C(\varphi, \psi_{\mathbf{p}})$ such that

$$p_{\tau} = e^{S_{|\tau|}\psi(\bar{\tau})} \leq C(\varphi, \psi_{\mathbf{p}}) e^{\alpha_{-}(\mathbf{p})S_{|\tau|}\varphi(\bar{\tau})}.$$

By the bounded distortion property (Lemma 2.13) and (4.3), there exists a uniform constant $D' \geq 1$ such that $e^{\alpha_{-}(\mathbf{p})S_{|\tau|}\varphi(\bar{\tau})} \leq D' d(x, \xi)^{\alpha_{-}(\mathbf{p})}$. We have thus shown (4.1). It remains to consider the case when $j > i + 1$. Using that $[x, y] \subset f_{\omega}^{-1}(\bar{U})$ we obtain as above

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(y)| \leq p_{\omega} \leq C(\varphi, \psi_{\mathbf{p}}) e^{\alpha_{-}(\mathbf{p})S_{|\omega|}\varphi(\bar{\omega})}.$$

By our assumption that $j > i + 1$ we have $d(x, y) \geq \text{diam}(f_{\omega}^{-1} f_{i+1}^{-1}(O))$, which implies that there exists a constant $D' \geq 1$ such that

$$e^{\alpha_{-}(\mathbf{p})S_{|\tau|}\varphi(\bar{\omega})} \leq D' d(x, y)^{\alpha_{-}(\mathbf{p})}.$$

The proof of the first assertion is complete. To prove the second assertion, it is sufficient to verify it for $C_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{N}^s$. To this end, we replace the estimate in (4.4) by the following estimate. By Lemma 3.2 and Lemma 3.5 there exists a constant $K' \geq 1$ (which only depends on \mathbf{p} and $|\mathbf{n}|$) such that

$$|C_{\mathbf{n}}(x) - C_{\mathbf{n}}(\xi)| \leq K' |\tau|^{|\mathbf{n}|} p_{\tau}.$$

Then proceeding as above, for every $\alpha < \alpha_{-}(\mathbf{p})$ there exists $D'' \geq 1$ such that

$$(4.5) \quad |C_{\mathbf{n}}(x) - C_{\mathbf{n}}(\xi)| \leq K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) e^{\alpha_{-}(\mathbf{p})S_{|\tau|}\varphi(\bar{\tau})} = K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) D'' d(x, \xi)^{\alpha} e^{(\alpha_{-}(\mathbf{p}) - \alpha)S_{|\tau|}\varphi(\bar{\tau})}.$$

Since $\alpha - \alpha_{-}(\mathbf{p}) < 0$, we have $K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) D'' e^{(\alpha_{-}(\mathbf{p}) - \alpha)S_{|\tau|}\varphi(\bar{\tau})} \rightarrow 0$ uniformly as $|\tau| \rightarrow \infty$. Hence, $C_{\mathbf{n}}$ is α -Hölder continuous. \square

Remark 4.2. By considering $\alpha = \alpha_{-}(\mathbf{p})$ in (4.5) and using that $|\tau| \asymp -\log d(x, \xi)$ we can derive that $C_{\mathbf{n}}(x) - C_{\mathbf{n}}(y) \ll (-\log d(x, y))^{|\mathbf{n}|} d(x, y)^{-\alpha_{-}}$. If $\alpha_{-} = 1$ this implies that $C_{\mathbf{n}}$ is convex Lipschitz ([MW86]).

In fact, this property was derived in [MW86] for the classical Takagi function, and for the higher order derivatives of the Lebesgue singular function in [AK06]. The property of convex Lipschitz can be used to prove that the graph of $C_{\mathbf{n}}$ has Hausdorff dimension one if $\alpha_- = 1$.

Recall that we have $\alpha_- \leq \delta := \dim_H(J)$ with equality if and only if $\alpha_- = \alpha_+$ ([Pes97, Chapter 21, see in particular, Figure 17b]).

Corollary 4.3. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Then for every $C \in \mathcal{T} \setminus \{0\}$ we have*

$$\alpha_- = \sup \{ \alpha \geq 0 \mid C \in \mathcal{C}^\alpha(\overline{\mathbb{R}}) \} \leq \delta.$$

The equality $\alpha_- = \alpha_+$ occurs if and only if $T_{\mathbf{p}} \in \mathcal{C}^\delta(\overline{\mathbb{R}})$.

Proof. The first assertion follows from Proposition 3.7 and Theorem 4.1. Now suppose that $\alpha_- = \alpha_+$. Hence, $\alpha_- = \delta$ and $T_{\mathbf{p}} \in \mathcal{C}^\delta(\overline{\mathbb{R}})$ by Theorem 4.1. Conversely, suppose that $T_{\mathbf{p}} \in \mathcal{C}^\delta(\overline{\mathbb{R}})$. Then, by the first assertion of Corollary 4.3, we have $\alpha_- \geq \delta$. Hence, $\alpha_- = \alpha_+$. \square

Corollary 4.4. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. If $\alpha_- < 1$ then, for each $1 > \alpha > \alpha_-$ there exists $\phi \in \mathcal{C}^\alpha(\overline{\mathbb{R}})$ such that $\|M_{\mathbf{p}}^n \phi\|_\alpha \rightarrow \infty$, as $n \rightarrow \infty$.*

Proof. Let $\phi \in \mathcal{C}^\alpha(\overline{\mathbb{R}})$ such that $\phi(\infty) = 1$ and $\phi(-\infty) = 0$. Then $\|M_{\mathbf{p}}^n \phi - T_{\mathbf{p}}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Now suppose for a contradiction that $\liminf_{n \geq 1} \|M_{\mathbf{p}}^n \phi\|_\alpha < \infty$. Then it follows that $T_{\mathbf{p}} \in \mathcal{C}^\alpha(\overline{\mathbb{R}})$ giving the desired contradiction to Theorem 1.4. \square

In the following, we will denote $C_{(n)}$ by C_n if $s = 1$ and $n \in \mathbb{N}_0$.

Proposition 4.5. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition.*

- (1) *If $\alpha_- = \alpha_+$ then $\mathcal{T} \cap \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}}) = \mathbb{R}T_{\mathbf{p}}$.*
- (2) *If $s = 1$ and each f_i has constant derivative, then $C_n \notin \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$ for every $n \geq 1$.*

Proof. We first prove (1). For $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^s$ we write $\mathbf{n} \leq \mathbf{m}$ if $n_1 \leq m_1, \dots, n_s \leq m_s$. Let $C = \sum_{\mathbf{n} \leq \mathbf{n}_{\max}} \beta_{\mathbf{n}} C_{\mathbf{n}}$ with $\beta_{\mathbf{n}_{\max}} \neq 0$ and $n := \sum_{i=1}^s (\mathbf{n}_{\max})_i \geq 1$. Our aim is to prove that $C \notin \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$. Let $a := \min J$ and $b := \max J$. Define for $k \geq 1$

$$m_i(k) := k^{((n+1)^{i-1})}, \quad 1 \leq i \leq s,$$

and for $\omega(k) := 1^{m_1(k)} 2^{m_2(k)} \dots s^{m_s(k)} \in I^{\sum_{i=1}^s m_i(k)}$ we let

$$x_k := (f_{\omega(k)})^{-1}(a), \quad y_k := (f_{\omega(k)})^{-1}(b).$$

Recall that, since $\alpha_- = \alpha_+$, we have $\alpha_- = \alpha_+ = \delta = \dim_H J$, and the potentials $\delta\phi$ and ψ are cohomologous. It follows that

$$\frac{|C(x_k) - C(y_k)|}{d(x_k, y_k)^{\alpha_-}} \asymp \frac{|C(x_k) - C(y_k)|}{P_{\omega(k)}}, \quad \text{as } k \rightarrow \infty.$$

Since $C_{\mathbf{n}}(a) - C_{\mathbf{n}}(b) = \mathbf{0}$, for $\mathbf{n} \neq \mathbf{0}$ and $C_{\mathbf{0}}(a) - C_{\mathbf{0}}(b) = 1$, we conclude that by Lemma 3.2,

$$\frac{C(x_k) - C(y_k)}{P_{\omega(k)}} = \sum_{\mathbf{n} \leq \mathbf{n}_{\max}} \beta_{\mathbf{n}} \left(A \left(\omega(k), \sum_{i=1}^s m_i(k) \right) U(a, b) \right)_{\mathbf{n}}.$$

By the combinatorial arguments in [JS17, Lemma 4.8] we have, as $k \rightarrow \infty$,

$$\frac{|C(x_k) - C(y_k)|}{P_{\omega(k)}} \asymp \left| \sum_{\mathbf{n} \leq \mathbf{n}_{\max}} \beta_{\mathbf{n}} \prod_{i=1}^s m_i(k)^{n_i} \right| \asymp \left| \sum_{\mathbf{n} \leq \mathbf{n}_{\max}} \beta_{\mathbf{n}} k^{\sum_{i=1}^s n_i (n+1)^{i-1}} \right| \asymp |\beta_{\mathbf{n}_{\max}}| k^{\sum_{i=1}^s (\mathbf{n}_{\max})_i (n+1)^{i-1}}.$$

It follows that $|C(x_k) - C(y_k)|/d(x_k, y_k)^{\alpha_-}$ is not bounded, as $k \rightarrow \infty$, which implies that $C \notin \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$. The proof of (1) is complete.

To prove (2) we can proceed along the same lines. First note that, by our assumptions on f_1 and f_2 , we have that φ depends only on the first coordinate. It follows from the median inequality that there exists $i \in I$ such that $\varphi(\bar{i})/\psi(\bar{i}) = \alpha_-$. Hence, for the sequences given by $x_k := (f_i)^{-k}(a)$ and $y_k := (f_i)^{-k}(b)$ we have as $k \rightarrow \infty$,

$$\frac{|C_n(x_k) - C_n(y_k)|}{d(x_k, y_k)^{\alpha_-}} \asymp \frac{|C_n(x_k) - C_n(y_k)|}{p_i^k} \asymp \left| \left(A(i^k, k)U(a, b) \right)_n \right| \asymp k^n,$$

which tends to infinity, as $k \rightarrow \infty$. Again this implies that $C_n \notin \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$. The proof is complete. \square

5. NON-DIFFERENTIABILITY

In this section we investigate the (non-)differentiability of the elements of \mathcal{T} . Note that $\alpha_- \leq \dim_H(J) \leq 1$. We denote by η the Lebesgue measure on $[0, 1]$.

Proposition 5.1. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition.*

- (1) *If $\alpha_- < 1$ then there exists a dense subset $E \subset J$ with $\dim_H(E) > 0$ such that, for every $C \in \mathcal{T} \setminus \{0\}$ and $x \in E$, C is not differentiable at x . If moreover $\alpha_+ < 1$ then $C \in \mathcal{T} \setminus \{0\}$ is nowhere differentiable on J .*
- (2) *If $\alpha_- < 1$ and $\dim_H(J) = 1$ then we have, for every $C \in \mathcal{T}$, $C'(x) = 0$ for η -a.e. $x \in J$, where η denotes the Lebesgue measure on \mathbb{R} .*
- (3) *If $\alpha_- = 1$ then $\alpha_+ = 1$ and $\dim_H(J) = 1$. Further, with $\mathbf{e}_k \in \mathbb{N}^s$ denoting the k -th unit vector in \mathbb{N}_0^s , $1 \leq k \leq s$, we have that $C_{\mathbf{e}_k}$ is nowhere differentiable on J . Moreover, if $s = 1$ then C_m is nowhere differentiable on J , for every $m \geq 1$.*

Proof. First assume that $\alpha_- < 1$. If $\alpha_- < \alpha_+$ then there exists $\alpha \in (\alpha_-, \alpha_+)$ with $\alpha < 1$. Then by Proposition 3.7 we have $\text{Höl}(C, x) \leq \alpha < 1$ for all $x \in E := \pi(\mathcal{F}(\alpha))$, and E has the desired properties. If moreover $\alpha_+ < 1$ then we have $\text{Höl}(C, x) \leq \alpha_+ < 1$ for every $x \in J$ by Proposition 3.7, which completes the proof of (1).

To prove (2), let $\alpha_- < 1$ and observe that $\dim_H(J) = 1$ implies $\alpha_- < \alpha_+$. Hence, t is strictly convex. Since $t(0) = 1$ and $t(1) \leq 0$, it follows that $-t'(0) > 1$. Thus, we have $\text{Höl}(C, x) = \int \psi d\mu_0 / \int \varphi d\mu_0 = -t'(0) > 1$ μ_0 -almost everywhere. Since μ_0 is equivalent to η , the assertion in (2) follows.

Finally, we turn to the proof of (3). For $1 \leq k \leq s$ let $\mathbf{e}_k \in \mathbb{N}_0^s$ denote the k -th unit vector in \mathbb{N}_0^s . Suppose for a contradiction that there exists $x \in J$ such that $C_{\mathbf{e}_k}$ is differentiable at x and let $\omega \in \Sigma$ such that $x = \pi(\omega)$. Let $a := \min J$ and $b := \max J$. Define $x_n := (f_{\omega|n})^{-1}(a)$ and $y_n := (f_{\omega|n})^{-1}(b)$, $n \geq 1$. We will verify that the sequence $(\gamma_n)_{n \geq 1}$ given by

$$\gamma_n := \frac{C_{\mathbf{e}_k}(x_n) - C_{\mathbf{e}_k}(y_n)}{d(x_n, y_n)}, \quad n \geq 1,$$

is not convergent. Since $x_n, y_n \rightarrow x$ we obtain the desired contradiction. To prove that (γ_n) is not convergent, recall that $\alpha_- = 1$ implies $\alpha_- = \alpha_+ = 1$ and that φ is cohomologous to ψ . Hence, we have

$$\frac{p_{\omega_n}}{d(x_n, y_n)} \asymp e^{S_n \psi(\omega) - S_n \varphi(\omega)} \asymp 1, \quad \text{as } n \rightarrow \infty.$$

Combining with the fact that $C_{\mathbf{e}_k}(a) - C_{\mathbf{e}_k}(b) = 0$, $T(a) - T(b) = 1$ and Lemma 3.2 we obtain

$$\gamma_n = \frac{C_{\mathbf{e}_k}(x_n) - C_{\mathbf{e}_k}(y_n)}{d(x_n, y_n)} \asymp \frac{(U(x_n, y_n))_{\mathbf{e}_k}}{p_{\omega_n}} = (A(\omega, n)U(a, b))_{\mathbf{e}_k} = A(\omega, n)_{\mathbf{e}_k, \mathbf{0}}(T(a) - T(b)) = A(\omega, n)_{\mathbf{e}_k, \mathbf{0}}.$$

Let $\delta_{i,j}$ denote the Dirac delta function, for $i, j \in I$. By the definition of the matrix cocycle A we have

$$\begin{aligned}
& A(\omega, n)_{\mathbf{e}_k, \mathbf{0}} \\
&= \sum_{i=1}^n A(\omega, 1)_{\mathbf{e}_k, \mathbf{e}_k} A(\sigma(\omega), 1)_{\mathbf{e}_k, \mathbf{e}_k} \cdots A(\sigma^{i-2}(\omega), 1)_{\mathbf{e}_k, \mathbf{e}_k} A(\sigma^{i-1}(\omega), 1)_{\mathbf{e}_k, \mathbf{0}} A(\sigma^i(\omega), 1)_{\mathbf{0}, \mathbf{0}} \cdots A(\sigma^{n-1}(\omega), 1)_{\mathbf{0}, \mathbf{0}} \\
(5.1) \\
&= \sum_{i=1}^n A(\sigma^{i-1}(\omega), 1)_{\mathbf{e}_k, \mathbf{0}} = \sum_{i=1}^n \left(\frac{\delta_{k, \omega_i}}{p_k} - \frac{\delta_{s+1, \omega_i}}{p_{s+1}} \right).
\end{aligned}$$

Hence, (γ_n) is not convergent which finishes the proof of the non-differentiability of $C_{\mathbf{e}_k}$.

Finally, we assume that $s = 1$ and consider C_m for $m \geq 1$. Proceeding as above, we have to verify that $A(\omega, n)_{m,0}$ is not convergent, as $n \rightarrow \infty$. If $m = 1$ this follows from (5.1). For $m > 1$, we will proceed by induction. It is easy to see that

$$\begin{aligned}
A(\omega, n+1)_{m,0} &= A(\omega, n)_{m,0} A(\sigma^n(\omega), 1)_{0,0} + A(\omega, n)_{m,1} A(\sigma^n(\omega), 1)_{1,0} \\
&= A(\omega, n)_{m,0} + A(\omega, n)_{m,1} A(\sigma^n(\omega), 1)_{1,0}.
\end{aligned}$$

Hence,

$$A(\omega, n+1)_{m,0} - A(\omega, n)_{m,0} = A(\omega, n)_{m,1} \cdot A(\sigma^n(\omega), 1)_{1,0} = A(\omega, n)_{m,1} \cdot \left(\frac{\delta_{1, \omega_{n+1}}}{p_1} - \frac{\delta_{2, \omega_{n+1}}}{p_2} \right).$$

Further, we have

$$\begin{aligned}
A(\omega, n)_{m,1} &= \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq n} \prod_{k=1}^{m-1} A(\sigma^{i_k-1}(\omega), 1)_{m-k+1, m-k} \\
&= \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq n} \prod_{k=1}^{m-1} \left(\frac{\delta_{1, \omega_{i_k}}}{p_1} - \frac{\delta_{2, \omega_{i_k}}}{p_2} \right) (m-k+1) \\
&= m \cdot \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq n} \prod_{k=1}^{m-1} \left(\frac{\delta_{1, \omega_{i_k}}}{p_1} - \frac{\delta_{2, \omega_{i_k}}}{p_2} \right) (m-k) \\
&= m \cdot A(\omega, n)_{m-1,0}.
\end{aligned}$$

Therefore,

$$A(\omega, n+1)_{m,0} - A(\omega, n)_{m,0} = m \cdot A(\omega, n)_{m-1,0} \cdot \left(\frac{\delta_{1, \omega_{n+1}}}{p_1} - \frac{\delta_{2, \omega_{n+1}}}{p_2} \right).$$

We conclude inductively that $A(\omega, n)_{m,0}$ is not convergent, as $n \rightarrow \infty$, for every $m \in \mathbb{N}$. The proof of (3) is complete. □

Remark 5.2. If $\alpha_- = \alpha_+ = 1$, then the nowhere-differentiability of elements of \mathcal{T} stated in Theorem 5.1 (3) shall be compared with the fact that $T_{\mathbf{p}}$ is a $\mathcal{C}^{1+\varepsilon}$ -diffeomorphism by Corollary 6.3.

6. CONJUGARIES BETWEEN INTERVAL MAPS

In this section we show how our results are related to interval conjugacy maps. This section is motivated by the results in [JKPS09] for conjugacy maps between expanding $C^{1+\varepsilon}$ maps on the unit interval with finitely many full branches. Note that in [JKPS09] it is always assumed that the Julia set J is equal to the unit interval.

For $\mathbf{p} \in (0, 1)^s$ we define the expanding linear maps $g_1, \dots, g_{s+1} : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ which are for $i \in I$ given by

$$g_i(x) := \frac{1}{p_i} \left(x - \sum_{1 \leq j < i} p_j \right).$$

Clearly, (g_1, \dots, g_{s+1}) satisfies our standing assumptions. Moreover, (g_1, \dots, g_{s+1}) satisfies the open set condition with open set $O := (0, 1)$ because $g_i^{-1}(O) = (\sum_{j < i} p_j, \sum_{j \leq i} p_j)$. The Julia set of (g_1, \dots, g_{s+1}) is equal to $[0, 1]$. We denote by $\pi_{\mathbf{p}} : \Sigma \rightarrow [0, 1]$ the coding map of the Julia set of (g_1, \dots, g_{s+1}) . Similarly, we define

$$g_{\mathbf{p}} : [0, 1] \rightarrow [0, 1], \quad g_{\mathbf{p}}(x) := g_i(x), \quad \text{where } i = \min \{j \in I \mid g_j(x) \in [0, 1]\}.$$

Note that $g_{\mathbf{p}}$ is the piecewise linear map on $[0, 1]$ with $(s+1)$ full branches and slopes given by $(1/p_i)_{i \in I}$.

Suppose that (f_1, \dots, f_{s+1}) satisfies the open set condition and suppose that $f_i^{-1}(O) \leq f_{i+1}^{-1}(O)$ for all $i \in I$. Denote the Julia set of (f_1, \dots, f_{s+1}) by J and its coding map by $\pi : \Sigma \rightarrow J$. We also define

$$f : J \rightarrow J, \quad f(x) := f_i(x), \quad \text{where } i = \min \{j \in I \mid f_j(x) \in J\}.$$

Further, we define

$$\Phi_{\mathbf{p}} : J \rightarrow [0, 1], \quad \Phi_{\mathbf{p}}(x) := \pi_{\mathbf{p}}(\omega) \text{ for some } \omega \in \pi^{-1}(x).$$

Note that $\Phi_{\mathbf{p}}$ is a well-defined, Borel measurable function satisfying $\Phi_{\mathbf{p}}(\max J) = 1$ and $\Phi_{\mathbf{p}}(\min J) = 0$. Let $\mathcal{E} := \{\omega \in \Sigma \mid \omega \text{ is eventually constant}\}$. For $x \in \pi(\Sigma \setminus \mathcal{E})$ we denote by $\pi^{-1}(x)$ the unique $\omega \in \Sigma \setminus \mathcal{E}$ such that $\pi(\omega) = x$. For $\omega \in \Sigma \setminus \mathcal{E}$ and $x = \pi(\omega)$ we have $f(x) = f_{\omega_1}(x)$ and thus,

$$\pi^{-1} \circ f(x) = \sigma(\omega).$$

Further, $g_{\mathbf{p}}(\pi_{\mathbf{p}}(\omega)) = g_{\omega_1}(\pi_{\mathbf{p}}(\omega))$ implies

$$\pi_{\mathbf{p}} \circ \sigma(\omega) = g_{\mathbf{p}} \circ \pi_{\mathbf{p}}(\omega).$$

Hence, for $x \in \pi(\Sigma \setminus \mathcal{E})$,

$$(6.1) \quad \Phi_{\mathbf{p}}(f(x)) = \pi_{\mathbf{p}} \circ \pi^{-1}(f(x)) = \pi_{\mathbf{p}} \circ \sigma(\omega) = g_{\mathbf{p}} \circ \pi_{\mathbf{p}}(\omega) = g_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)).$$

If J is an interval, the following lemma appears implicitly in [JKPS09, Proof of Proposition 1.4].

Lemma 6.1. *For every $\mathbf{p} \in (0, 1)^s$ we have that $\Phi_{\mathbf{p}} = T_{\mathbf{p}}|_J$.*

Proof. We proceed in two steps. First, we will show that $\Phi_{\mathbf{p}}(x) = T_{\mathbf{p}}(x)$ for $x \in \pi(\Sigma \setminus \mathcal{E})$. Let $\tilde{\Phi}_{\mathbf{p}} : \overline{\mathbb{R}} \rightarrow [0, 1]$ denote a bounded Borel measurable extension of $\Phi_{\mathbf{p}}$ such that $\tilde{\Phi}_{\mathbf{p}}(y) = 0$ for $y \in [-\infty, \min J]$, and $\tilde{\Phi}_{\mathbf{p}}(y) = 1$ for $y \in [\max J, +\infty]$. Let $\omega \in \Sigma \setminus \mathcal{E}$. For $i < \omega_1$ we have $f_i(\pi(\omega)) \geq \max J$, and for $i > \omega_1$ we have $f_i(\pi(\omega)) \leq \min J$. Since $\Phi_{\mathbf{p}}(\max J) = 1$ and $\Phi_{\mathbf{p}}(\min J) = 0$, the equality in (6.1) yields for $x = \pi(\omega)$

$$M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i \in I} p_i \tilde{\Phi}_{\mathbf{p}}(f_i(x)) = \sum_{i < \omega_1} p_i + p_{\omega_1} \cdot \Phi_{\mathbf{p}}(f(x)) + \sum_{i > \omega_1} 0 = \sum_{i < \omega_1} p_i + p_{\omega_1} \cdot g_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)).$$

Since $x = \pi(\omega)$ and $\omega \in \Sigma \setminus \mathcal{E}$, we have $g_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)) = g_{\omega_1}(\Phi_{\mathbf{p}}(x))$. Hence, we have for every $x \in \pi(\Sigma \setminus \mathcal{E})$,

$$M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i < \omega_1} p_i + p_{\omega_1} \cdot g_{\omega_1}(\Phi_{\mathbf{p}}(x)) = \sum_{i < \omega_1} p_i + p_{\omega_1} \cdot \frac{1}{p_{\omega_1}} \left(\Phi_{\mathbf{p}}(x) - \sum_{i < \omega_1} p_i \right) = \tilde{\Phi}_{\mathbf{p}}(x).$$

Further, for every $x \in [-\infty, \min J] \cup [\max J, +\infty]$ we have $M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \tilde{\Phi}_{\mathbf{p}}(x)$. Let $E := \pi(\Sigma \setminus \mathcal{E}) \cup [-\infty, \min J] \cup [\max J, +\infty]$. Since $f_i(E) \subset E$ for every $i \in I$, we can show inductively that for every $x \in E$ and $n \in \mathbb{N}$,

$$M_{\mathbf{p}}^n(\tilde{\Phi}_{\mathbf{p}})(x) = M_{\mathbf{p}}(M_{\mathbf{p}}^{n-1}\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i \in I} p_i (M_{\mathbf{p}}^{n-1}\tilde{\Phi}_{\mathbf{p}})(f_i(x)) = \sum_{i \in I} p_i \tilde{\Phi}_{\mathbf{p}}(f_i(x)) = \tilde{\Phi}_{\mathbf{p}}(x).$$

By (2.2) we conclude that for $x \in \pi(\Sigma \setminus \mathcal{E})$,

$$\tilde{\Phi}_{\mathbf{p}}(x) = \lim_{n \rightarrow \infty} M_{\mathbf{p}}^n \tilde{\Phi}_{\mathbf{p}}(x) = T_{\mathbf{p}}(x) \tilde{\Phi}_{\mathbf{p}}(\infty) + (1 - T_{\mathbf{p}}(x)) \tilde{\Phi}_{\mathbf{p}}(-\infty) = T_{\mathbf{p}}(x).$$

This completes the proof of $\Phi_{\mathbf{p}}(x) = T_{\mathbf{p}}(x)$ for $x \in \pi(\Sigma \setminus \mathcal{E})$. Now, let $\omega \in \mathcal{E}$ and $x = \pi(\omega)$. Let $(\omega^{(n)}) \subset \Sigma \setminus \mathcal{E}$ such that $(\omega_1^{(n)}, \dots, \omega_n^{(n)}) = (\omega_1, \dots, \omega_n)$ and $x_n := \pi(\omega^{(n)})$, $n \geq 1$. By the continuity of π with respect to the word metric, we have $x_n \rightarrow x$ as $n \rightarrow \infty$. By the definition of $\Phi_{\mathbf{p}}$ we have $\Phi_{\mathbf{p}}(x_n) = \pi_{\mathbf{p}}(\omega^{(n)})$ and $\Phi_{\mathbf{p}}(x) = \pi_{\mathbf{p}}(\omega)$. So, by the continuity of $\pi_{\mathbf{p}}$ with respect to the word metric, we have $\Phi_{\mathbf{p}}(x_n) \rightarrow \Phi_{\mathbf{p}}(x)$ as $n \rightarrow \infty$. Since $\Phi_{\mathbf{p}}(x_n) = T_{\mathbf{p}}(x_n)$ by the first part of the proof, and since $T_{\mathbf{p}}$ is continuous by Theorem 2.4, we can conclude that $\Phi_{\mathbf{p}}(x) = T_{\mathbf{p}}(x)$. The proof is complete. \square

Theorem 6.2 ([JKPS09, Theorem 1.2]). *Suppose that $(f_i)_{i \in I}$ be $\mathcal{C}^{1+\varepsilon}$ -diffeomorphisms satisfying the open set condition. Let J be an interval. Then for every $\mathbf{p} \in (0, 1)^s$ the following rigidity dichotomy holds.*

- (1) If $\alpha_- = \alpha_+ (= 1)$ then $T_{\mathbf{p}}$ is a $\mathcal{C}^{1+\varepsilon}$ -diffeomorphism.
- (2) If $\alpha_- < \alpha_+$ then $T_{\mathbf{p}}' \equiv 0$ η -a.e., $T_{\mathbf{p}} \in \mathcal{C}^{\alpha_-}(\overline{\mathbb{R}})$ and the set of non-differentiability points of $T_{\mathbf{p}}$ has positive Hausdorff dimension.

Combining the previous theorem with the fact that $T_{\mathbf{p}} \in \mathcal{C}^{\alpha_-(\mathbf{p})}(J)$ (see Theorem 4.1) we obtain the following corollary from Lemma 6.1, Theorem 6.2 and Corollary 4.3. Recall that we have $\alpha_-(\mathbf{p}) \leq \dim_H(J)$ with equality if and only if $\alpha_- = \alpha_+$.

Corollary 6.3. *Suppose that $(f_i)_{i \in I}$ be $\mathcal{C}^{1+\varepsilon}$ -diffeomorphisms satisfying the open set condition. Let $\delta := \dim_H(J)$. For $\mathbf{p} \in (0, 1)^s$ we have $\alpha_-(\mathbf{p}) = \alpha_+(\mathbf{p})$ if and only if $T_{\mathbf{p}}$ is $\mathcal{C}^{\delta}(\overline{\mathbb{R}})$. If $\alpha_-(\mathbf{p}) = 1$ then $\alpha_- = \alpha_+ = 1$, $J = \overline{0}$ and $T_{\mathbf{p}}$ is a $\mathcal{C}^{1+\varepsilon}$ -diffeomorphism.*

7. APPENDIX: CONTRACTIONS NEAR INFINITY

The property that $(f_i)_{i \in I}$ is contracting near infinity depends on the choice of the metric d . We will show that, by modifying the $(f_i)_{i \in I}$ near infinity, we can always assume that an expanding family $(f_i)_{i \in I}$ is contracting near infinity with respect to a metric d which is strongly equivalent to the Euclidean metric on compact subsets of \mathbb{R} .

We consider the metric d on $\overline{\mathbb{R}}$ induced by the bijection

$$h : \overline{\mathbb{R}} \rightarrow [-1, 1], \quad h(x) := \frac{x}{1 + |x|}.$$

Namely, we set

$$d(x, y) := |h(x) - h(y)|.$$

Note that the metric d generates the topology of the two-point compactification of \mathbb{R} . Moreover, d is strongly equivalent to the Euclidean metric on compact subsets of \mathbb{R} .

Now suppose that $(f_i)_{i \in I}$ is expanding with expansion rate $\lambda > 1$. We can take g_i , $i \in I$, such that $g_i = f_i$ in a neighbourhood of $\mathbb{R} \setminus (V_+ \cup V_-)$ and

$$g_i'(x) \rightarrow \lambda, \quad \text{as } x \rightarrow \pm\infty.$$

Then we can prove the following lemma.

Lemma 7.1. *There exist neighbourhoods V^{\pm} of $\pm\infty$ such that $\text{Lip}(g_i|_{V^{\pm}}) < 1$ for each $i \in I$.*

Proof. Let $i \in I$. For $x, y \in \mathbb{R}$ if x, y are close to ∞ or x, y are close to $-\infty$, then we have

$$(7.1) \quad \frac{d(g_i(x), g_i(y))}{d(x, y)} = \frac{|(1 + |x|)(1 + |y|)|}{(1 + |g_i(x)|)(1 + |g_i(y)|)} \left| \frac{g_i(x) - g_i(y)}{x - y} \right|.$$

By our assumptions we have $\limsup_{u \rightarrow \infty} u/g_i(u) \leq \lambda^{-1}$ and $\lim_{u \rightarrow \infty} g_i'(u) \rightarrow \lambda$. Hence, it follows from (7.1) that for x, y sufficiently large,

$$\frac{d(g_i(x), g_i(y))}{d(x, y)} < 1.$$

It remains to consider the case when $x = \infty$. The case when $x = -\infty$ is similar and therefore omitted. For y sufficiently large such that $g_i(y) \geq (\lambda - \eta)y$, for some η with $\lambda - \eta > 1$, we have

$$\frac{d(g_i(\infty), g_i(y))}{d(\infty, y)} = \frac{1 + y}{1 + g_i(y)} \leq \frac{1 + y}{1 + (\lambda - \eta)y} < 1.$$

Hence, there exists a neighbourhood V^+ of $+\infty$ such that $\text{Lip}(g_i|_{V^+}) < 1$. \square

REFERENCES

- [AK06] P. C. Allaart and K. Kawamura, *Extreme values of some continuous nowhere differentiable functions*, Math. Proc. Cambridge Philos. Soc. **140** (2006), no. 2, 269–295. MR 2212280
- [All17] P. C. Allaart, *Differentiability and Hoelder spectra of a class of self-affine functions*, arXiv:1707.07376 (2017).
- [BKK16] B. Barany, G. Kiss, and I. Kolossvary, *Pointwise regularity of parameterized affine zipper fractal curves*, arXiv:1608.04558 (2016).
- [Fal03] K. Falconer, *Fractal geometry*, second ed., John Wiley & Sons Inc., Hoboken, NJ, 2003, Mathematical foundations and applications. MR MR2118797 (2006b:28001)
- [HY84] M. Hata and M. Yamaguti, *The Takagi function and its generalization*, Japan J. Appl. Math. **1** (1984), no. 1, 183–199. MR 839313
- [ITM50] C. T. Ionescu Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non complètement continues*, Ann. of Math. (2) **52** (1950), 140–147. MR 0037469
- [JKPS09] T. Jordan, M. Kesseböhmer, M. Pollicott, and B. O. Stratmann, *Sets of nondifferentiability for conjugacies between expanding interval maps*, Fund. Math. **206** (2009), 161–183. MR 2576266
- [JS15] J. Jaerisch and H. Sumi, *Multifractal formalism for expanding rational semigroups and random complex dynamical systems*, Nonlinearity **28** (2015), 2913–2938.
- [JS17] ———, *Pointwise Hölder exponents of the complex analogues of the Takagi function in random complex dynamics*, Adv. Math. **313** (2017), 839–874.
- [Kat76] T. Kato, *Perturbation theory for linear operators*, second ed., Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132. MR MR0407617 (53 #11389)
- [KS08] M. Kesseböhmer and B. O. Stratmann, *Fractal analysis for sets of non-differentiability of Minkowski's question mark function*, J. Number Theory **128** (2008), no. 9, 2663–2686. MR MR2444218
- [KS09] ———, *Hölder -differentiability of Gibbs distribution functions*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 2, 489–503. MR MR2525939
- [Lju83] M. J. Ljubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 351–385. MR 741393
- [MU03] R. D. Mauldin and M. Urbański, *Graph directed Markov systems*, Cambridge Tracts in Mathematics, vol. 148, Cambridge, 2003.
- [MW86] R. D. Mauldin and S. C. Williams, *On the Hausdorff dimension of some graphs*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 793–803. MR 860394
- [Pat97] N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math. **19** (1997), no. 4, 486–513. MR 1479016
- [Pes97] Y. B. Pesin, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997, Contemporary views and applications. MR MR1489237 (99b:58003)
- [Roc70] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR MR0274683 (43 #445)
- [Sch99] J. Schmeling, *On the completeness of multifractal spectra*, Ergodic Theory Dynam. Systems **19** (1999), no. 6, 1595–1616. MR 1738952

- [SS91] T. Sekiguchi and Y. Shiota, *A generalization of Hata-Yamaguti's results on the Takagi function*, Japan J. Indust. Appl. Math. **8** (1991), no. 2, 203–219. MR 1111613
- [Sum11] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (1) (2011), no. 102, 50–112.
- [Sum13] ———, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math. **245** (2013), 137–181.
- [Wal82] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982. MR MR648108 (84e:28017)

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