Random complex dynamics and singular functions on the complex plane

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Abstract: We investigate the random complex dynamics and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$. We see that in the random complex dynamics, the chaos easily disappears. We investigate the iteration of the transition operator M acting on the space of continuous functions on $\hat{\mathbb{C}}$. It turns out that under certain conditions, each finite linear combination φ of unitary eigenvectors of M can be regarded as a complex analogue of the devil's staircase. By using Birkhoff's ergodic theorem and potential theory, we investigate the non-differentiability and the pointwise Hölder exponent of φ. The contents of this presentation are included in my preprint "Random complex dynamics and semigroups of holomorphic maps" which is available from my webpage above or from http://arxiv.org/abs/0812.4483. Date: May 18, 2009.

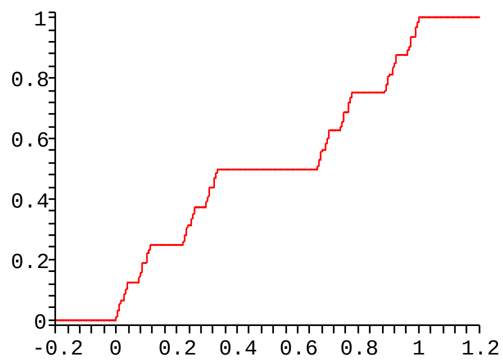
1 Introduction

First, we consider the random dynamics on $\mathbb{R}$.

- Let $h_1(x) = 3x$ and $h_2(x) = 3(x-1)+1$ $(x \in \mathbb{R})$.
- We take an initial value $x \in \mathbb{R}$, and at every step, we choose the map h_1 with probability $1/2$ and h_2 with probability $1/2$, and map the point under the chosen map h_j.
- Let $T_{+\infty}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$.

Then, $T_{+\infty}$ is continuous on $\mathbb{R}$, varies only on the Cantor middle third set (which is a thin fractal set), and monotone.

$T_{+\infty}$ is called **the devil's staircase**. This is a typical example of singular functions.



We will consider a complex analogue of this story.

2 Preliminaries

Definition 2.1.

- We denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ the Riemann sphere and denote by d the spherical distance on $\hat{\mathbb{C}}$.
- We set Rat$:=\{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-const. rational map$\}$ endowed with the distance η defined by $\eta(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$.
- We set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g$ is a polynomial map$, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.
- Note that Rat and $\mathcal{P}$ are semigroups where the semigroup operation is functional composition.
- A subsemigroup G of Rat is called a **rational semigroup**.
- A subsemigroup G of $\mathcal{P}$ is called a **polynomial semigroup**.

Definition 2.2. Let G be a rational semigroup.

- We set $F(G) := \{z \in \hat{\mathbb{C}} \mid \exists$ nbd U of z s.t. G is equicontinuous on $U\}$.
  This $F(G)$ is called the **Fatou set** of G.
- We set $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the **Julia set** of G.
- If G is generated by $\{h_1, \ldots, h_m\}$ as a semigroup, we write $G = \langle h_1, \ldots, h_m \rangle$.

Lemma 2.3. *Let G be a rational semigroup. Then $F(G)$ is open and $J(G)$ is compact. Moreover, for each $h \in G$,*

$$h(F(G)) \subset F(G) \text{ and } h^{-1}(J(G)) \subset J(G).$$

However, the equality $h^{-1}(J(G)) = J(G)$ does not hold in general.

Remark 2.4. The fact we do not have $h^{-1}(J(G)) = J(G)$ is the difficulty in this theory. However, we 'utilize' this fact for the study of the random complex dynamics.

Lemma 2.5. *If a rational semigroup G is generated by a compact subset Λ of* Rat, *then $J(G) = \bigcup_{g \in \Lambda} g^{-1}(J(G))$. In particular, if $G = \langle h_1, \ldots, h_m \rangle$, then $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$. This property of $J(G)$ is called the* **backward self-similarity**.

Definition 2.6. For a topological space X, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X endowed with the weak topology.

Remark 2.7. If X is a compact metric space, then $\mathfrak{M}_1(X)$ is a compact metric space.

From now on, we take a $\tau \in \mathfrak{M}_1(\text{Rat})$ and we consider the (i.i.d.) random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in$ Rat according to τ.

Definition 2.8. Let $\tau \in \mathfrak{M}_1(\text{Rat})$.

(1) We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ endowed with the sup. norm $\| \ \|_\infty$.

(2) Let $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ be the operator defined by $M_\tau(\varphi)(z) := \int_{\text{Rat}} \varphi(g(z)) \, d\tau(g)$, where $\varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}$.

(3) We set $C(\hat{\mathbb{C}})^* := \{\rho : C(\hat{\mathbb{C}}) \to \mathbb{C} \mid \rho \text{ is linear and continuous}\}$ endowed with the weak topology.

(4) Let $M_\tau^* : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$ be the dual of M_τ. That is, $M_\tau^*(\rho)(\varphi) := \rho(M_\tau(\varphi))$ for each $\rho \in C(\hat{\mathbb{C}})^*$ and for each $\varphi \in C(\hat{\mathbb{C}})$.
Note that $M_\tau^*(\mathfrak{M}_1(\hat{\mathbb{C}})) \subset \mathfrak{M}_1(\hat{\mathbb{C}})$.

(5) We set
$F_{meas}(\tau) := \{\mu \in \mathfrak{M}_1(\hat{\mathbb{C}}) \mid \exists$ nbd B of μ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ s.t.
$\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous on $B\}$.

(6) We set $J_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$.

(7) Let $\mathcal{U}_\tau$ be the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is 1.

(8) Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$.

(9) Let $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\text{Rat})^{\mathbb{N}})$.

(10) Let G_τ be the rational semigroup generated by $\operatorname{supp} \tau$.

The following is the key to investigating the random complex dynamics.

Definition 2.9. Let G be a rational semigroup. We set

$$J_{\ker}(G) := \bigcap_{h \in G} h^{-1}(J(G)).$$

This is called the **kernel Julia set** of G.

Remark 2.10. $J_{\ker}(G)$ is a compact subset of $J(G)$. Moreover, for each $h \in G$, $h(J_{\ker}(G)) \subset J_{\ker}(G)$.

Lemma 2.11. *Let Γ be a compact subset of $\mathcal{P}$. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset U of $\hat{\mathbb{C}}$ such that $\{f_0 + c \mid c \in U\} \subset \Gamma$, then the polynomial semigroup G generated by Γ satisfies that $J_{\ker}(G) = \emptyset$.*

The above lemma implies that from a point of view, for most $\tau \in \mathfrak{M}_1(\mathcal{P})$ with compact support, we have $J_{\ker}(G_\tau) = \emptyset$.

Question 2.12. What happens if $J_{\ker}(G_\tau) = \emptyset$?

3 Results

Theorem 3.1 (Theorem A, Cooperation Principle). *Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be s.t.* $\mathrm{supp}\,\tau$ *is compact. Suppose $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, we have all of the following.*

(1) $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$ (Chaos disappears!).

(2) $\mathcal{B}_{0,\tau}$ *is a closed subspace of* $C(\hat{\mathbb{C}})$ *and* $C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \mathcal{B}_{0,\tau}$.

(3) $\dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$.

(4) *For each $\varphi \in \mathcal{U}_\tau$ and for each connected component U of $F(G_\tau)$, $\varphi|_U$ is constant.*

(5) *For $\forall z \in \hat{\mathbb{C}}$, $\exists \mathcal{A}_z \subset (\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ with the following property.*
   - $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, $\exists \delta = \delta(z, \gamma) > 0$ *s.t.* $\mathrm{diam}\gamma_n \cdots \gamma_1(B(z, \delta)) \to 0$ *as* $n \to \infty$, *where* diam *denotes the diameter w.r.t. the spherical distance.*

(6) *For $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$, the 2-dim. Leb. meas. of* $J_\gamma := \{z \in \hat{\mathbb{C}} \mid \{\gamma_n \circ \cdots \circ \gamma_1\}_{n \in \mathbb{N}}$ *is not equiconti. on $\forall$nbd of $z\}$ is equal to zero.*

(7) *There exist at least one and at most finitely many minimal sets of G_τ in $\hat{\mathbb{C}}$, where we say that a non-empty compact subset K of $\hat{\mathbb{C}}$ is a minimal set of G_τ in $\hat{\mathbb{C}}$ if K is minimal in $\{L \subset \hat{\mathbb{C}} \mid \emptyset \neq L$ is compact, $\forall g \in G_\tau, g(L) \subset L\}$ w.r.t. $\subset$.*

(8) *Let L_τ be the union of minimal sets of G_τ. Then $\forall z \in \hat{\mathbb{C}}$ $\exists \mathcal{C}_z \subset (\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ s.t. $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{C}_z$, $d(\gamma_n \cdots \gamma_1(z), L_\tau) \to 0$ as $n \to \infty$.*

Remark 3.2. Theorem A describes new phenomena which cannot hold in the usual iteration dynamics of a single $g \in \mathrm{Rat}$ with $\deg(g) \geq 2$.

Definition 3.3. Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. We set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1(\mathcal{P}^{\mathbb{N}})$. For any $z \in \hat{\mathbb{C}}$, we set

$$T_{\infty,\tau}(z) := \tilde{\tau}(\{\gamma \in \mathcal{P}^{\mathbb{N}} \mid \gamma_n \circ \cdots \circ \gamma_1(z) \to \infty \text{ as } n \to \infty\}),$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots)$.

$T_{\infty,\tau}(z)$ is the probability of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value $z \in \hat{\mathbb{C}}$ with respect to the random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathcal{P}$ according to τ.

Theorem 3.4. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ be such that* $\operatorname{supp} \tau$ *is compact. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then, $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous on the whole $\hat{\mathbb{C}}$. Moreover, for each connected component U of $F(G_\tau)$, $T_{\infty,\tau}|_U$ is constant. Furthermore, $M_\tau(T_{\infty,\tau}) = T_{\infty,\tau}$ and $T_{\infty,\tau} \in \mathcal{U}_\tau$.*

Remark 3.5. Such a function $T_{\infty,\tau}$ is called

a devil's coliseum

provided that $T_{\infty,\tau} \not\equiv 1$. In fact, $T_{\infty,\tau}$ is a complex analogue of the devil's staircase. For the graph of $T_{\infty,\tau}$, see Figure 2 (page 11) and Figure 3 (page 12).

We now consider the non-differentiability of non-const. elements $\varphi \in \mathcal{U}_\tau$ at $J(G_\tau)$.

Theorem 3.6 (Theorem B). *Let $h_1, h_2 \in \mathcal{P}$ and let $0 < p_1, p_2 < 1$ with $p_1 + p_2 = 1$. We set $\tau := \sum_{i=1}^{2} p_i \delta_{h_i} \in \mathfrak{M}_1(\mathcal{P})$. Let*
$P(G_\tau) := \overline{\bigcup_{h \in G_\tau} \{\text{all critical values of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$ $(\subset \hat{\mathbb{C}})$.
We assume that

(a) G_τ *is hyperbolic* (i.e. $P(G_\tau) \subset F(G_\tau)$),
(b) $h_1^{-1}(J(G_\tau)) \cap h_2^{-1}(J(G_\tau)) = \emptyset$, *and*
(c) $\exists z \in \mathbb{C}$ s.t. $\bigcup_{h \in G_\tau} \{h(z)\}$ *is bounded in* $\mathbb{C}$.

Then, we have all of the following statements (1), (2), (3).

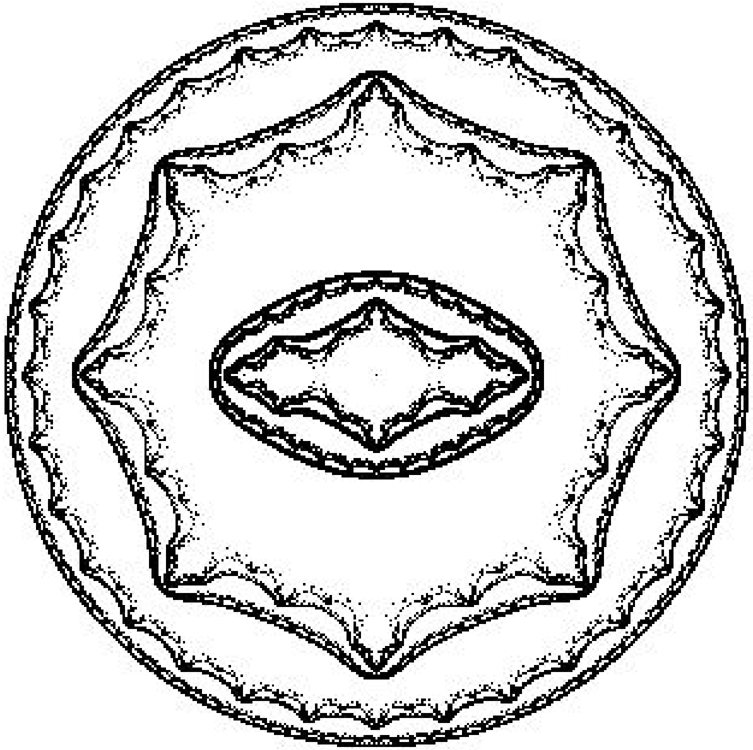
(1) $J_{\ker}(G_\tau) = \emptyset$, $T_{\infty,\tau} \in \mathcal{U}_\tau$ *and* $T_{\infty,\tau}$ *is non-constant.*
(2) $\dim_H(J(G_\tau)) < 2$, *where* $\dim_H$ *denotes the Hausdorff dimension w.r.t. Euclidean dist.*
(3) $\exists$ *dense* $A \subset J(G_\tau)$ *with* $\dim_H(A) > 0$ s.t. $\forall z \in A$, $\forall$*non-const.* $\varphi \in \mathcal{U}_\tau$,

the pointwise Hölder exponent of φ at z

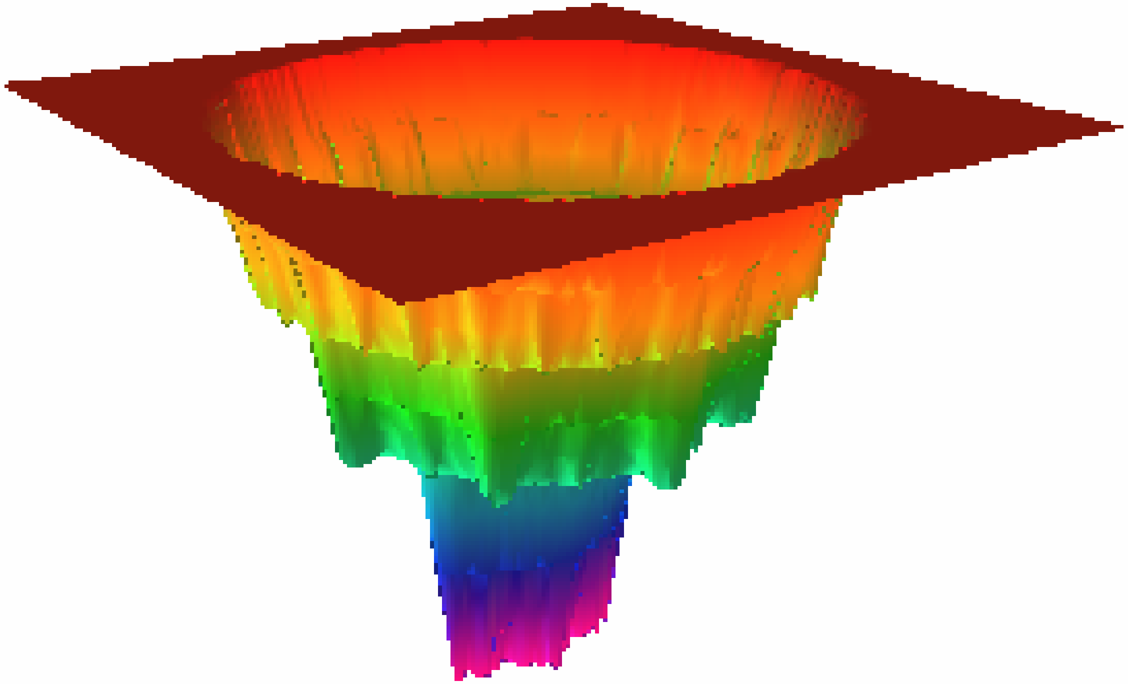
$$:= \inf\{\alpha \in \mathbb{R} \mid \overline{\lim}_{y \to z} \frac{|\varphi(y) - \varphi(z)|}{|y - z|^\alpha} = \infty\}$$
$$= \frac{\text{entropy of } (p_1, p_2)}{\text{"averaged Lyapunov exponent"}} < 1$$

and φ *is not differentiable at* z. *(av. Lyap. exp. is represented by* $p_i, \deg(h_i)$, *and an integral related to the random Green's functions.)*

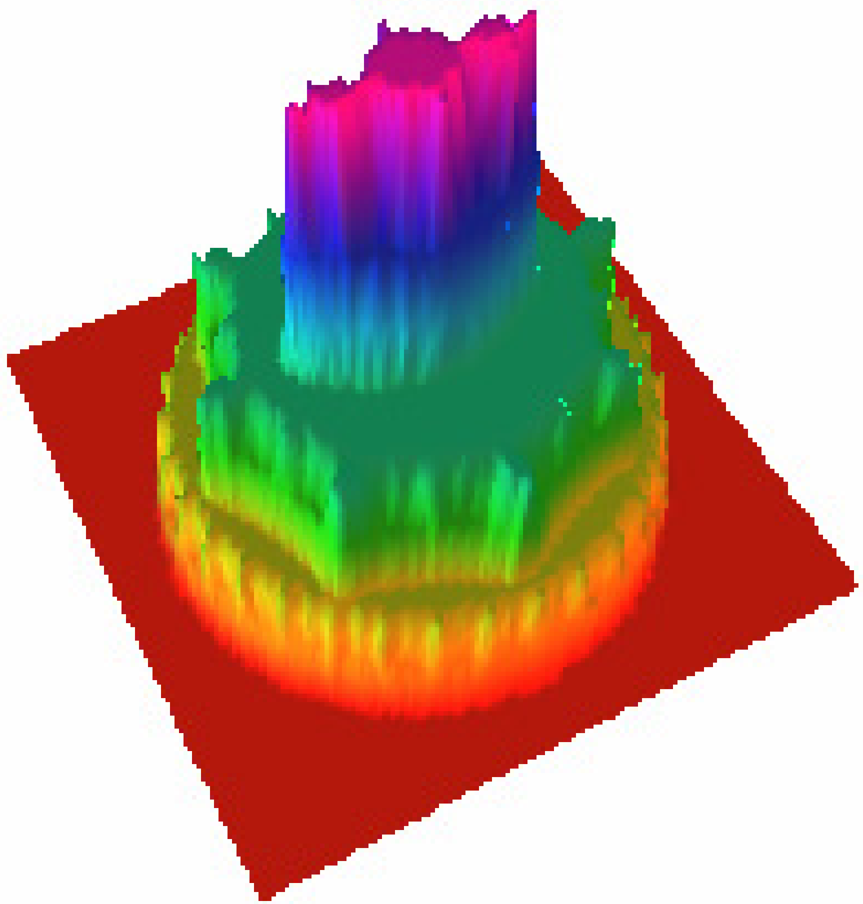
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2.\ G := \langle h_1, h_2 \rangle.\ G \in \mathcal{G}_{dis}.$
The figure of $J(G)$. $\sharp \text{Con}(J(G)) > \aleph_0$.



$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2,\ \tau := \frac{1}{2}\delta_{h_1} + \frac{1}{2}\delta_{h_2}.$
The graph of $z \mapsto T_{\tau,\infty}(z)$.
(Devil's Coliseum (Complex analogue of devil's staircase).)



The graph of $z \mapsto 1 - T_{\tau,\infty}(z)$.



We describe the detail of statement (3) of Theorem B.

- Let $\Gamma = \{h_1, h_2\}$ and for each $(\gamma, y) = ((\gamma_1, \gamma_2, \ldots), y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, we set

$$\mathcal{G}_\gamma(y) := \lim_{n\to\infty} \frac{1}{\deg(\gamma_n \circ \cdots \circ \gamma_1)} \max\{\log|\gamma_n \circ \cdots \circ \gamma_1(y)|, 0\}$$

- For each $\gamma \in \Gamma^{\mathbb{N}}$, let $\mu_\gamma := dd^c\mathcal{G}_\gamma \in \mathfrak{M}_1(J_\gamma) \subset \mathfrak{M}_1(J(G_\tau))$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. We set $\mu := \int_{\Gamma^{\mathbb{N}}} \mu_\gamma \; d\tilde{\tau}(\gamma) \in \mathfrak{M}_1(J(G_\tau))$.
- For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, let $\Omega(\gamma) := \sum_c \mathcal{G}_\gamma(c)$, where c runs over all critical points of γ_1 in $\mathbb{C}$.

Then, regarding (3) of Theorem B, we have the following.

(a) $\dim_H(A) \geq \dim_H(\mu) = \frac{\sum_{i=1}^2 p_i \log\deg(h_i) - \sum_{i=1}^2 p_i \log p_i}{\sum_{i=1}^2 p_i \log\deg(h_i) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma) \; d\tilde{\tau}(\gamma)}$.

(b) $\dfrac{\textit{entropy of } (p_1,p_2)}{\textit{“averaged Lyapunov exponent”}}$
$= \frac{-\sum_{i=1}^2 p_i \log p_i}{\sum_{i=1}^2 p_i \log\deg(h_i) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma) \; d\tilde{\tau}(\gamma)}$.

Remark 3.7. In the proof of statement (3) of Theorem B, we use Birkhoff's ergodic theorem **(ergodic theory)**, Koebe distortion theorem **(function theory)**, and the random Green's functions and calculation of Lyapunov exponent **(potential theory)**.

4 Example

Proposition 4.1. *Let $h_1 \in \mathcal{P}$ be hyperbolic.*

- *Suppose that $K(h_1)$ is connected and* int$K(h_1) \neq \emptyset$, *where* $K(h_1) := \{z \in \mathbb{C} \mid \{h_1^n(z)\}_{n \in \mathbb{N}}$ *is bounded*$\}$.
- *Let $b \in$* int$K(h_1)$.
- *Let $d \in \mathbb{N}$ with $d \geq 2$ be* s.t. $(\deg(h_1), d) \neq (2, 2)$.

Then $\exists c > 0$ s.t. $\forall a \in \mathbb{C}$ *with* $0 < |a| < c$,
setting $h_2(z) = a(z - b)^d + b$,
$\{h_1, h_2\}$ *satisfies the assumption of Theorem B, i.e.,*

(a) $G = \langle h_1, h_2 \rangle$ *is hyperbolic,*
(b) $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, *and*
(c) $\exists z \in \mathbb{C}$ s.t. $\bigcup_{h \in G} \{h(z)\}$ *is bounded in* $\mathbb{C}$.

5 Summary

- We simultaneously develop the theory of random complex dynamics and that of the dynamics of semigroups of holomorphic maps.

- Both fields are related to each other very deeply.

- In the random complex dynamics, the chaos easily disappears, due to the cooperation of the generator maps.

- In the random complex dynamics, if the chaos disappears, then in the limit stage, singular functions on the complex plane (devil's coliseums) appear. They are complex analogues of the devil's staircase or Lebesgue's singular functions. Thus, even if the chaos disappears, we still have a kind of complexity. In fact, the chaos disappears in "C^0" sense, but the chaos may remain in "C^1" sense. In this context, the pointwise Hölder exponent of the complex singular functions are important.

- Under certain conditions, the pointwise Hölder exponent of the complex singular functions are represented by the ratio of entropy of the given probability and the averaged Lyapunov exponent, which can be calculated by the probability, degree of generators, and an integral related to the random Green's functions.

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