

# MULTIFRACTAL FORMALISM FOR EXPANDING RATIONAL SEMIGROUPS AND RANDOM COMPLEX DYNAMICAL SYSTEMS

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**ABSTRACT.** We consider the multifractal formalism for the dynamics of semigroups of rational maps on the Riemann sphere and random complex dynamical systems. We elaborate a multifractal analysis of level sets given by quotients of Birkhoff sums with respect to the skew product associated with a semigroup of rational maps. Applying these results, we perform a multifractal analysis of the Hölder regularity of limit state functions of random complex dynamical systems.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $\text{Rat}$  denote the set of all non-constant rational maps on the Riemann sphere  $\widehat{\mathbb{C}}$ . A subsemigroup of  $\text{Rat}$  with semigroup operation being functional composition is called a *rational semigroup*. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([HM96]), who were interested in the role of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional module spaces for discrete groups, and by F. Ren's group ([GR96]), who studied such semigroups from the perspective of random dynamical systems. The first study of random complex dynamics was given by J. E. Fornæss and N. Sibony ([FS91]). For the motivations and the recent studies on random complex dynamics, see the second author's work [Sum11, Sum13].

The study of multifractals goes back to the work of [Man74, FP85, HJK<sup>+</sup>86] which was motivated by statistical physics. In this paper, we perform a multifractal analysis of level sets given by quotients of Birkhoff sums with respect to the skew product associated with a rational semigroup. We recommend [PW97, Pes97] for a similar kind of multifractal analysis for conformal repellers.

One of our main motivations to develop the multifractal formalism for rational semigroups is to apply our results to random complex dynamical systems. The multifractal formalism allows us to investigate level sets of a prescribed Hölder regularity of limit state functions (linear combinations of unitary eigenfunctions of transition operators) of random complex dynamical systems. In this way, our multifractal analysis exhibits a refined gradation between chaos and order for random complex dynamical systems, which has been recently studied by the second author in [Sum11, Sum13]. We remark that this paper is the first one in which the multifractal formalism is applied to the study of limit state functions of random complex dynamical systems.

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Also, we note that under certain conditions such a limit state function is continuous on  $\widehat{\mathbb{C}}$  but varies precisely on the Julia set of the associated rational semigroup, which is a thin fractal set. From this point of view, this study is deeply related to both random dynamics and fractal geometry.

Throughout this paper, we will always assume that  $I$  is a finite set. An element  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  is called a *multi-map*. We say that  $f = (f_i)_{i \in I}$  is *exceptional* if  $\text{card}(I) = 1$  and  $\deg(f_i) = 1$  for  $i \in I$ . Throughout, we always assume that  $f$  is non-exceptional. Let  $f = (f_i)_{i \in I}$  be a multi-map and let  $G = \langle f_i : i \in I \rangle$ , where  $\langle f_i : i \in I \rangle$  denotes the rational semigroup generated by  $(f_i)_{i \in I}$ , i.e.,  $G = \{f_{i_1} \circ \dots \circ f_{i_n} \mid n \in \mathbb{N}, i_1, \dots, i_n \in I\}$ . Let  $(p_i)_{i \in I}$  be a probability vector and let  $\tau := \sum_{i \in I} p_i \delta_{f_i}$  where  $\delta_{f_i}$  refers to the Dirac measure supported on  $f_i$ . We always assume that  $0 < p_i < 1$  for each  $i \in I$ . We consider the i.i.d. random dynamical system associated with  $\tau$ , i.e., at every step we choose a map  $f_i$  with probability  $p_i$ . We denote by  $C(\widehat{\mathbb{C}})$  the Banach space of complex-valued continuous functions on  $\widehat{\mathbb{C}}$  endowed with the supremum norm. The transition operator  $M_\tau : C(\widehat{\mathbb{C}}) \rightarrow C(\widehat{\mathbb{C}})$  is the bounded linear operator given by  $M_\tau(\varphi)(z) := \sum_{i \in I} \varphi(f_i(z)) p_i$ , for each  $\varphi \in C(\widehat{\mathbb{C}})$  and  $z \in \widehat{\mathbb{C}}$ . A function  $\rho \in C(\widehat{\mathbb{C}})$  with  $\rho \not\equiv 0$  is called a unitary eigenfunction of  $M_\tau$  if there exists  $u \in \mathbb{C}$  with  $|u| = 1$  such that  $M_\tau(\rho) = u\rho$ . Let  $U_\tau$  be the space of all linear combinations of unitary eigenfunctions of  $M_\tau : C(\widehat{\mathbb{C}}) \rightarrow C(\widehat{\mathbb{C}})$ . To investigate the regularity of the elements in  $U_\tau$  we consider the following quantities.

**Definition 1.1.** Let  $\rho : \widehat{\mathbb{C}} \rightarrow \mathbb{C}$  be a bounded function and let  $z \in \widehat{\mathbb{C}}$ . We set

$$Q_*(\rho, z) := \liminf_{r \rightarrow 0} \frac{\log Q(\rho, z, r)}{\log r}, \quad Q^*(\rho, z) := \limsup_{r \rightarrow 0} \frac{\log Q(\rho, z, r)}{\log r} \quad \text{and} \quad Q(\rho, z) := \lim_{r \rightarrow 0} \frac{\log Q(\rho, z, r)}{\log r},$$

where  $Q(\rho, z, r)$  is for  $r > 0$  given by

$$Q(\rho, z, r) := \sup_{y \in B(z, r)} |\rho(y) - \rho(z)|.$$

Moreover, we define for each  $\alpha \in \mathbb{R}$  the corresponding level sets

$$R_*(\rho, \alpha) := \{y \in \widehat{\mathbb{C}} : Q_*(\rho, y) = \alpha\}, \quad R^*(\rho, \alpha) := \{y \in \widehat{\mathbb{C}} : Q^*(\rho, y) = \alpha\}$$

and

$$R(\rho, \alpha) := \{y \in \widehat{\mathbb{C}} : Q(\rho, y) = \alpha\}.$$

Let  $d$  denote the spherical distance on  $\widehat{\mathbb{C}}$ . The pointwise Hölder exponent  $\text{Höl}(\rho, z)$  of  $\rho$  at  $z$  is given by

$$\text{Höl}(\rho, z) := \sup \left\{ \beta \in [0, \infty) : \limsup_{y \rightarrow z, y \neq z} \frac{|\rho(y) - \rho(z)|}{d(y, z)^\beta} < \infty \right\} \in [0, \infty].$$

The level set  $H(\rho, \alpha)$  with prescribed Hölder exponent  $\alpha \in \mathbb{R}$  is given by

$$H(\rho, \alpha) := \{y \in \widehat{\mathbb{C}} : \text{Höl}(\rho, y) = \alpha\}.$$

In fact, we will show in Lemma 5.1 that  $\text{Höl}(\rho, z) = Q_*(\rho, z)$  for every  $z \in \widehat{\mathbb{C}}$ . We refer to Section 5 for the proof of this fact and for further properties of the quantities introduced in Definition 1.1.

We proceed by introducing the necessary preliminaries to state our first main result. The *Fatou set*  $F(G)$  and the *Julia set*  $J(G)$  of a rational semigroup  $G$  are given by

$$F(G) := \{z \in \widehat{\mathbb{C}} : G \text{ is normal in a neighbourhood of } z\} \quad \text{and} \quad J(G) := \widehat{\mathbb{C}} \setminus F(G).$$

If  $G$  is a rational semigroup generated by a single map  $g \in \text{Rat}$ , then we write  $G = \langle g \rangle$ . Moreover, for a single map  $g \in \text{Rat}$ , we set  $F(g) := F(\langle g \rangle)$  and  $J(g) := J(\langle g \rangle)$ .

Let  $f = (f_i)_{i \in I}$  be a multi-map. The skew product associated with the multi-map  $f = (f_i)_{i \in I}$  is given by

$$\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow I^{\mathbb{N}} \times \widehat{\mathbb{C}}, \quad \tilde{f}(\omega, z) := (\sigma(\omega), f_{\omega_1}(z)),$$

where  $\sigma : I^{\mathbb{N}} \rightarrow I^{\mathbb{N}}$  denotes the left-shift map given by  $\sigma(\omega_1, \omega_2, \dots) := (\omega_2, \omega_3, \dots)$ , for  $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$ . We say that a multi-map  $f = (f_i)_{i \in I}$  is *expanding* if the associated skew product  $\tilde{f}$  is expanding along fibres on the Julia set  $J(\tilde{f})$  (see Definition 2.4).

We say that  $\psi = (\psi_i)_{i \in I}$  is a *Hölder family associated with the multi-map*  $f = (f_i)_{i \in I}$  if  $\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R}$  is Hölder continuous for each  $i \in I$ , where  $G = \langle f_i : i \in I \rangle$  and  $J(G)$  is equipped with the metric inherited from the spherical distance  $d$  on  $\widehat{\mathbb{C}}$ . Note that  $\cup_{i \in I} f_i^{-1}(J(G)) = J(G)$  ([Sum00, Lemma 2.4]). If it is clear from the context with which multi-map  $\psi$  is associated, then we simply say that  $\psi$  is a Hölder family. For a Hölder family  $\psi = (\psi_i)_{i \in I}$ , we define  $\tilde{\psi} : J(\tilde{f}) \rightarrow \mathbb{R}$  given by  $\tilde{\psi}(\omega, z) := \psi_{\omega_1}(z)$ , for all  $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$  and  $z \in f_{\omega_1}^{-1}(J(G))$ , and for each  $n \in \mathbb{N}$  we denote by  $S_n \tilde{\psi} : J(\tilde{f}) \rightarrow \mathbb{R}$  the Birkhoff sum of  $\tilde{\psi}$  with respect to  $\tilde{f}$  given by  $S_n \tilde{\psi} := \sum_{i=0}^{n-1} \tilde{\psi} \circ \tilde{f}^i$ .

For an expanding multi-map  $f = (f_i)_{i \in I}$ , let  $\zeta = (\zeta_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be the Hölder family given by  $\zeta_i(z) := -\log \|f_i'(z)\|$  for each  $i \in I$  and  $z \in f_i^{-1}(J(G))$ , where  $\|\cdot\|$  denotes the norm of the derivative with respect to the spherical metric on  $\widehat{\mathbb{C}}$ . Let  $\pi_{\widehat{\mathbb{C}}} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  denote the canonical projection. We define the level sets  $\mathcal{F}(\alpha, \psi)$ , which are for  $\alpha \in \mathbb{R}$  given by

$$\mathcal{F}(\alpha, \psi) := \pi_{\widehat{\mathbb{C}}}(\tilde{\mathcal{F}}(\alpha, \psi)), \quad \text{where } \tilde{\mathcal{F}}(\alpha, \psi) := \left\{ x \in J(\tilde{f}) : \lim_{n \rightarrow \infty} \frac{S_n \tilde{\psi}(x)}{S_n \zeta(x)} = \alpha \right\}.$$

The (Hausdorff-) dimension spectrum  $l$  of  $(f, \psi)$  is given by

$$l(\alpha) := \dim_H(\mathcal{F}(\alpha, \psi)), \quad \text{for } \alpha \in \mathbb{R}.$$

The range of the multifractal spectrum is given by

$$\alpha_-(\psi) := \inf \{ \alpha \in \mathbb{R} : \mathcal{F}(\alpha, \psi) \neq \emptyset \} \quad \text{and} \quad \alpha_+(\psi) := \sup \{ \alpha \in \mathbb{R} : \mathcal{F}(\alpha, \psi) \neq \emptyset \}.$$

The *free energy function* for  $(f, \psi)$  is the unique function  $t : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathcal{P}(\beta \tilde{\psi} + t(\beta) \zeta, \tilde{f}) = 0$  for each  $\beta \in \mathbb{R}$ , where  $\mathcal{P}(\cdot, \tilde{f})$  denotes the topological pressure with respect to  $\tilde{f}$  ([Wal82]). The number  $t(0)$  is also referred to as the *critical exponent*  $\delta$  of  $f$ . The convex conjugate of  $t$  ([Roc70, Section 12]) is given by

$$t^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}, \quad t^*(c) := \sup_{\beta \in \mathbb{R}} \{ \beta c - t(\beta) \}, \quad c \in \mathbb{R}.$$

We say that  $f = (f_i)_{i \in I}$  satisfies the *separation condition* if  $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G)) = \emptyset$  for all  $i, j \in I$  with  $i \neq j$ , where  $G := \langle f_i : i \in I \rangle$ . Note that in this case, under the assumption  $J(G) \neq \emptyset$ , for any probability vector  $(p_i)_{i \in I} \in (0, 1)^I$ , setting  $\tau := \sum_{i \in I} p_i \delta_{f_i}$ , we have that (1)  $1 \leq \dim_{\mathbb{C}}(U_{\tau}) < \infty$  and (2) there exists a bounded linear operator  $\pi_{\tau} : C(\widehat{\mathbb{C}}) \rightarrow U_{\tau}$  such that, for each  $\varphi \in C(\widehat{\mathbb{C}})$ , we have  $\|M_{\tau}^n(\varphi - \pi_{\tau}(\varphi))\| \rightarrow 0$ , as  $n$  tends to infinity (see [Sum11, Theorem 3.15]). If an element  $\rho \in U_{\tau}$  is non-constant, then  $\rho$  is continuous on  $\widehat{\mathbb{C}}$  and the set of varying points of  $\rho$  is equal to the thin fractal set  $J(G)$  (for the figure of the graph of such a function  $\rho$ , see [Sum11]). Such  $\rho$  is considered as a complex analogue of a devil's staircase or Lebesgue's singular function. Some of such functions  $\rho$  are called devil's coliseums (see [Sum11]). Our first main result shows that, for every  $\rho \in U_{\tau}$  non-constant, the level sets  $R_*$ ,  $R$ ,  $R^*$  and  $H$  satisfy the multifractal formalism.

**Theorem 1.2** (Theorem 5.3). *Let  $f = (f_i)_{i \in I}$  be an expanding multi-map and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $f$  satisfies the separation condition. Let  $(p_i)_{i \in I} \in (0, 1)^I$  be a probability vector and let  $\tau := \sum_{i \in I} p_i \delta_{f_i}$ . Suppose there exists a non-constant function belonging to  $U_{\tau}$ . Let  $\rho \in C(\widehat{\mathbb{C}})$  be a non-constant function*

belonging to  $U_\tau$ . Let  $\psi = (\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be given by  $\psi_i(z) := \log p_i$ . Let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Then we have the following.

- (1) There exists a number  $a \in (0, 1)$  such that  $\rho : \widehat{\mathbb{C}} \rightarrow \mathbb{C}$  is  $a$ -Hölder continuous and  $a \leq \alpha_-(\psi)$ .
- (2) We have  $\alpha_+(\psi) = \sup\{\alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset\}$  and  $\alpha_-(\psi) = \inf\{\alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset\}$ . Moreover,  $H$  can be replaced by  $R_*$ ,  $R$  or  $R^*$ .
- (3) Let  $\alpha_\pm := \alpha_\pm(\psi)$ . If  $\alpha_- < \alpha_+$  then we have for each  $\alpha \in (\alpha_-, \alpha_+)$ ,

$$\dim_H(R^*(\rho, \alpha)) = \dim_H(R_*(\rho, \alpha)) = \dim_H(R(\rho, \alpha)) = \dim_H(H(\rho, \alpha)) = -t^*(-\alpha) > 0.$$

Moreover,  $s(\alpha) := -t^*(-\alpha)$  defines a real analytic and strictly concave positive function on  $(\alpha_-, \alpha_+)$  with maximum value  $\delta > 0$ . Also,  $s'' < 0$  on  $(\alpha_-, \alpha_+)$ .

- (4) (a) For each  $i \in I$  we have  $\deg(f_i) \geq 2$ . Moreover, we have  $\alpha_- = \alpha_+$  if and only if there exists an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$ , complex numbers  $(a_i)_{i \in I}$  and  $\lambda \in \mathbb{R}$  such that for all  $i \in I$ ,

$$\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)} \quad \text{and} \quad \log \deg(f_i) = \lambda \log p_i.$$

- (b) If  $\alpha_- = \alpha_+$  then we have

$$R^*(\rho, \alpha_-) = R_*(\rho, \alpha_-) = R(\rho, \alpha_-) = H(\rho, \alpha_-) = J(G),$$

where  $\dim_H(J(G)) = \delta > 0$  and  $R^*(\rho, \alpha) = R_*(\rho, \alpha) = R(\rho, \alpha) = H(\rho, \alpha) = \emptyset$  for all  $\alpha \neq \alpha_-$ .

We denote by  $C^\alpha(\widehat{\mathbb{C}}) := \{\varphi : \widehat{\mathbb{C}} \rightarrow \mathbb{C} \mid \|\varphi\|_\alpha < \infty\}$  the Banach space of  $\alpha$ -Hölder continuous functions on  $\widehat{\mathbb{C}}$  endowed with the  $\alpha$ -Hölder norm

$$\|\varphi\|_\alpha := \sup_{z \in \widehat{\mathbb{C}}} |\varphi(z)| + \sup_{x, y \in \widehat{\mathbb{C}}, x \neq y} |\varphi(x) - \varphi(y)| / d(x, y)^\alpha.$$

Under the assumptions of Theorem 1.2 and some additional conditions, we have  $\alpha_- < 1$  (see the Remark in Section 6). In this case Theorem 1.2 implies that for each  $\alpha \in (\alpha_-, 1)$  the iteration of the transition operator  $M_\tau$  does not behave well on the Banach space  $C^\alpha(\widehat{\mathbb{C}})$ , i.e., there exists an element  $\varphi \in C^\alpha(\widehat{\mathbb{C}})$  such that  $\|M_\tau^n(\varphi)\|_\alpha \rightarrow \infty$  as  $n \rightarrow \infty$ . It means that, regarding the iteration of the transition operator  $M_\tau$  on functions spaces, even though the chaos disappears on  $C(\widehat{\mathbb{C}})$  and  $C^\alpha(\widehat{\mathbb{C}})$ , we still have a kind of complexity (or chaos) on the space  $C^\alpha(\widehat{\mathbb{C}})$  for each  $\alpha \in (\alpha_-, 1)$ . Thus, in random complex dynamical systems we sometimes have a kind of gradation between chaos and order. Theorem 1.2 can be seen as a refinement of this gradation. In Section 6 we give many examples to which we can apply Theorem 1.2.

In order to show our first main result, we prove the general multifractal formalism for level sets given by quotients of Birkhoff sums with respect to the skew product associated with a semigroup of rational maps. We say that a multi-map  $f = (f_i)_{i \in I}$  satisfies the *open set condition* if there exists a non-empty open set  $U$  in  $\widehat{\mathbb{C}}$  such that  $\{f_i^{-1}(U) : i \in I\}$  consists of pairwise disjoint subsets of  $U$ . Note that the open set condition is weaker than the separation condition, since the higher iterates of an expanding multi-map satisfying the separation condition also satisfy the open set condition.

Our second main result shows that, for an expanding multi-map satisfying only the open set condition and for an arbitrary Hölder family  $\psi$ , the level sets  $\mathcal{F}(\alpha, \psi)$  satisfy the multifractal formalism.

**Theorem 1.3** (Theorem 4.5). *Let  $f = (f_i)_{i \in I}$  be an expanding multi-map which satisfies the open set condition. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Suppose that there exists  $\gamma \in \mathbb{R}$  such that  $\mathcal{P}(\gamma\tilde{\psi}, \tilde{f}) = 0$  and suppose that  $\alpha_-(\psi) < \alpha_+(\psi)$ . Then (1) the Hausdorff dimension spectrum  $l$  of  $(f, \psi)$  is real analytic and strictly concave on*

$(\alpha_-(\psi), \alpha_+(\psi))$  with maximal value  $\delta$ , (2)  $l'' < 0$  on  $(\alpha_-(\psi), \alpha_+(\psi))$ , and (3) for  $\alpha \in (\alpha_-(\psi), \alpha_+(\psi))$  we have that

$$l(\alpha) = -t^*(-\alpha) > 0.$$

*Remark.* Note that in Theorem 1.3 we only assume that the multi-map  $f$  satisfies the open set condition, which does not imply that  $f$  satisfies the separation condition. In fact, there are many 2-generator expanding polynomial semigroups satisfying the open set condition for which the Julia set is connected (see [Sum14]). Theorem 1.3 can be applied even to such semigroups. Moreover, let us remark that, since each map of the generator system  $(f_i)_{i \in I}$  is not injective in general, we need much more efforts in the proof of Theorem 1.3 than in the case of contracting iterated function systems, even if the open set condition is assumed.

*Remark.* Note that, in general, we cannot replace the Hausdorff dimension by the box-counting dimension in Theorem 1.3. In fact, if  $\alpha = -t'(\beta)$  for some  $\beta \in \mathbb{R}$ , then we have that  $v_\beta(\mathcal{F}(\alpha, \psi)) = 1$  and  $\text{supp}(v_\beta) = J(G)$  by Lemmas 3.13 and 3.6, where  $v_\beta$  is given in Definition 3.7. Hence,  $\mathcal{F}(\alpha, \psi)$  is dense in  $J(G)$ , which implies that  $\dim_B(\mathcal{F}(\alpha, \psi)) = \dim_B(J(G))$ , where  $\dim_B$  refers to the box-counting dimension.

The results stated in Proposition 1.4 and Corollary 1.5 below follow from the general theory without assuming the open set condition (see Theorem 4.5 (1) and (2)). If each potential  $\psi_i$  is constant, then we have the following criterion for a non-trivial multifractal spectrum.

**Proposition 1.4** (Proposition 4.6). *Let  $f = (f_i)_{i \in I}$  be an expanding multi-map and let  $G = \langle f_i : i \in I \rangle$ . (We do not assume the open set condition.) Suppose that  $\deg(f_{i_0}) \geq 2$  for some  $i_0 \in I$ . Let  $(c_i)_{i \in I}$  be a sequence of negative numbers and let the Hölder family  $\psi = (\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be given by  $\psi_i(z) = c_i$  for each  $z \in f_i^{-1}(J(G))$ . Then we have  $\alpha_-(\psi) = \alpha_+(\psi)$  if and only if there exist an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$ , complex numbers  $(a_i)_{i \in I}$  and  $\lambda \in \mathbb{R}$  such that for all  $i \in I$ ,*

$$\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)} \quad \text{and} \quad \log \deg(f_i) = \lambda c_i.$$

*Remark.* Let us point out the relation between Proposition 1.4 and rigidity results in thermodynamic formalism. There are several equivalent formulations to characterize when the multifractal spectrum degenerates. Namely, the equality  $\alpha_-(\psi) = \alpha_+(\psi)$  is equivalent to each of the following statements: (1) The equilibrium states  $\tilde{\mu}_0$  and  $\tilde{\mu}_\gamma$  coincide. (2) The graph of the free energy function  $t$  is a straight line. (3) There exists a continuous function  $u : J(\tilde{f}) \rightarrow \mathbb{R}$  such that  $\delta \tilde{\zeta} - \gamma \tilde{\psi} = u - u \circ \tilde{f}$ . For precise statement and the proof, we refer to Proposition 3.10 and the proofs of Theorem 4.5 and Proposition 4.6.

*Remark.* The proof of Proposition 1.4 makes use of a rigidity result of Zdunik ([Zdu90]) for the classical iteration of a single rational map. We give an extension of this result to rational semigroups. We emphasize that the map  $\varphi$  in Proposition 1.4 is independent of  $i \in I$ .

An interesting special case is given by the Lyapunov spectrum of an expanding multi-map  $f = (f_i)_{i \in I}$ . The Lyapunov spectrum is given by the level sets  $\mathcal{L}(\alpha)$ ,  $\alpha \in \mathbb{R}$ , where we define

$$\mathcal{L}(\alpha) := \mathcal{F}(\alpha, -1) = \left\{ z \in \widehat{\mathbb{C}} : \exists \omega \in I^{\mathbb{N}} \text{ such that } (\omega, z) \in J(\tilde{f}) \text{ and } \lim_{n \rightarrow \infty} \frac{n}{\log \|(f_{\omega_n} \circ \dots \circ f_{\omega_1})'(z)\|} = \alpha \right\}.$$

We say that  $f = (f_i)_{i \in I}$  has trivial Lyapunov spectrum if there exists  $\alpha_0 \in \mathbb{R}$  such that  $\mathcal{L}(\alpha) = \emptyset$  if  $\alpha \neq \alpha_0$ .

**Corollary 1.5** (Lyapunov spectrum). *Let  $f = (f_i)_{i \in I}$  be an expanding multi-map. Suppose that  $\deg(f_{i_0}) \geq 2$  for some  $i_0 \in I$ . Then  $f$  has trivial Lyapunov spectrum if and only if there exist an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$  and complex numbers  $(a_i)_{i \in I}$  such that  $\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_{i_0})}$ .*

The paper is organised as follows. In Section 2 we collect the necessary preliminaries on the dynamics of expanding rational semigroups. In Section 3 we recall basic facts from thermodynamic formalism for expanding dynamical systems in the framework of the skew product associated with an expanding rational semigroup. In Section 4 we investigate the local dimension of conformal measures supported on subsets of the Julia set of rational semigroups satisfying the open set condition and we prove a multifractal formalism for Hölder continuous potentials in Theorem 4.5. In Section 5 we apply the multifractal formalism to investigate the Hölder regularity of linear combinations of unitary eigenfunctions of transition operators in random complex dynamics. Finally, examples of our results are given in Section 6.

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## 2. PRELIMINARIES

Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be a multi-map and let  $G = \langle f_i : i \in I \rangle$ . For  $n \in \mathbb{N}$  and  $(\omega_1, \omega_2, \dots, \omega_n) \in I^n$ , we set  $f_{(\omega_1, \omega_2, \dots, \omega_n)} := f_{\omega_n} \circ f_{\omega_{n-1}} \circ \dots \circ f_{\omega_1}$ . For  $\omega \in I^{\mathbb{N}}$  we set  $\omega_n := (\omega_1, \omega_2, \dots, \omega_n)$  and we define

$$F_\omega := \left\{ z \in \widehat{\mathbb{C}} : (f_{\omega_n})_{n \in \mathbb{N}} \text{ is normal in a neighbourhood of } z \right\} \text{ and } J_\omega := \widehat{\mathbb{C}} \setminus F_\omega.$$

The *skew product* associated with  $f = (f_i)_{i \in I}$  is given by

$$\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow I^{\mathbb{N}} \times \widehat{\mathbb{C}}, \quad \tilde{f}(\omega, z) := (\sigma(\omega), f_{\omega_1}(z)),$$

where  $\sigma : I^{\mathbb{N}} \rightarrow I^{\mathbb{N}}$  denotes the left-shift map given by  $\sigma(\omega_1, \omega_2, \dots) := (\omega_2, \omega_3, \dots)$ , for  $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$ .

For each  $\omega \in I^{\mathbb{N}}$ , we set  $J^\omega := \{\omega\} \times J_\omega$  and we set

$$J(\tilde{f}) := \overline{\bigcup_{\omega \in I^{\mathbb{N}}} J^\omega}, \quad F(\tilde{f}) := (I^{\mathbb{N}} \times \widehat{\mathbb{C}}) \setminus J(\tilde{f}),$$

where the closure is taken with respect to the product topology on  $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ . Let  $\pi_1 : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow I^{\mathbb{N}}$  and  $\pi_{\widehat{\mathbb{C}}} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  denote the canonical projections. We refer to [Sum00, Proposition 3.2] for the proof of the following proposition.

**Proposition 2.1.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and let  $G = \langle f_i : i \in I \rangle$ . Let  $\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow I^{\mathbb{N}} \times \widehat{\mathbb{C}}$  be the skew product associated with  $f$ . Then we have the following.*

- (1)  $\tilde{f}(J^\omega) = J^{\sigma\omega}$  and  $(\tilde{f}|_{\pi_1^{-1}(\omega)})^{-1}(J^{\sigma\omega}) = J^\omega$ , for each  $\omega \in I^{\mathbb{N}}$ .
- (2)  $\tilde{f}(J(\tilde{f})) = J(\tilde{f})$ ,  $\tilde{f}^{-1}(J(\tilde{f})) = J(\tilde{f})$ ,  $\tilde{f}(F(\tilde{f})) = F(\tilde{f})$ ,  $\tilde{f}^{-1}(F(\tilde{f})) = F(\tilde{f})$ .
- (3) Let  $G = \langle f_i : i \in I \rangle$  and suppose that  $\text{card}(J(G)) \geq 3$ . Then we have  $J(\tilde{f}) = \bigcap_{n \in \mathbb{N}_0} \tilde{f}^{-n}(I^{\mathbb{N}} \times J(G))$  and  $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) = J(G)$ . Here,  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

**Definition 2.2.** Let  $G$  be a rational semigroup and let  $z \in \widehat{\mathbb{C}}$ . The backward orbit  $G^-(z)$  of  $z$  and the set of exceptional points  $E(G)$  are defined by  $G^-(z) := \bigcup_{g \in G} g^{-1}(z)$  and  $E(G) := \left\{ z \in \widehat{\mathbb{C}} : \text{card}(G^-(z)) < \infty \right\}$ . We say that a set  $A \subset \widehat{\mathbb{C}}$  is  $G$ -backward invariant, if  $g^{-1}(A) \subset A$ , for each  $g \in G$ , and we say that  $A$  is  $G$ -forward invariant, if  $g(A) \subset A$ , for each  $g \in G$ .

The following is proved in [HM96] (see also [Sum00, Lemma 2.3], [Sta12]).

**Lemma 2.3.** *The following holds for a rational semigroup  $G$ .*

- (a)  $F(G)$  is  $G$ -forward invariant and  $J(G)$  is  $G$ -backward invariant.
- (b) If  $\text{card}(J(G)) \geq 3$ , then  $J(G)$  is a perfect set.
- (c) If  $\text{card}(J(G)) \geq 3$ , then  $\text{card}(E(G)) \leq 2$ .
- (d) If  $z \in \widehat{\mathbb{C}} \setminus E(G)$ , then  $J(G) \subset \overline{G^-(z)}$ . In particular, if  $z \in J(G) \setminus E(G)$  then  $\overline{G^-(z)} = J(G)$ .
- (e) If  $\text{card}(J(G)) \geq 3$ , then  $J(G)$  is the smallest closed set containing at least three points which is  $G$ -backward invariant.
- (f) If  $\text{card}(J(G)) \geq 3$ , then

$$J(G) = \overline{\left\{ z \in \widehat{\mathbb{C}} : z \text{ is a repelling fixed point of some } g \in G \right\}} = \overline{\bigcup_{g \in G} J(g)},$$

where the closure is taken in  $\widehat{\mathbb{C}}$ .

For a holomorphic map  $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  and  $z \in \widehat{\mathbb{C}}$ , the norm of the derivative of  $h$  at  $z \in \widehat{\mathbb{C}}$  with respect to the spherical metric is denoted by  $\|h'(z)\|$ .

**Definition 2.4.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and let  $\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \rightarrow I^{\mathbb{N}} \times \widehat{\mathbb{C}}$  denote the associated skew product. For each  $n \in \mathbb{N}$  and  $(\omega, z) \in J(\tilde{f})$ , we set  $(\tilde{f}^n)'(\omega, z) := (f_{\omega|_n})'(z)$ . We say that  $\tilde{f}$  is expanding along fibers if  $J(\tilde{f}) \neq \emptyset$  and if there exist constants  $C > 0$  and  $\lambda > 1$  such that for all  $n \in \mathbb{N}$ ,

$$\inf_{(\omega, z) \in J(\tilde{f})} \|(\tilde{f}^n)'(\omega, z)\| \geq C\lambda^n,$$

where  $\|(\tilde{f}^n)'(\omega, z)\|$  denotes the norm of the derivative of  $f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}$  at  $z$  with respect to the spherical metric.  $f = (f_i)_{i \in I}$  is called expanding if  $\tilde{f}$  is expanding along fibers. Also,  $G = \langle f_i : i \in I \rangle$  is called expanding if  $f = (f_i)_{i \in I}$  is expanding.

*Remark 2.5.* It follows from Proposition 2.8 below that, for a rational semigroup  $G = \langle f_i : i \in I \rangle$ , the notion of expandingness is independent of the choice of the generator system.

The following lemma was proved in [Sum01, Theorem 2.14] (see also Proposition 2.8 below).

**Lemma 2.6.** *If  $f = \langle f_i : i \in I \rangle$  is expanding, then we have  $J(\tilde{f}) = \bigcup_{\omega \in I^{\mathbb{N}}} J^\omega$ .*

**Definition 2.7.** A rational semigroup  $G$  is hyperbolic if  $P(G) \subset F(G)$ , where  $P(G)$  denotes the postcritical set of  $G$  given by

$$P(G) := \overline{\bigcup_{g \in G} \text{CV}(g)}$$

and where  $\text{CV}(g)$  denotes the set of critical values of  $g$ .

*Remark.* Let  $G = \langle f_i : i \in I \rangle$ . Since  $P(G) = \overline{\bigcup_{g \in G \cup \{\text{id}\}} g(\bigcup_{i \in I} \text{CV}(f_i))}$ , we have that  $P(G)$  is  $G$ -forward invariant.

In the next proposition we give necessary and sufficient conditions for a rational semigroup to be expanding. We refer to [Sum98] for the proofs. For  $g \in \text{Aut}(\widehat{\mathbb{C}})$ , we say that  $g$  is *loxodromic* if  $g$  has exactly two fixed points, for which the modulus of the multipliers is not equal to one.

**Proposition 2.8.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and  $G = \langle f_i : i \in I \rangle$ .*

- (1) *If  $f$  is expanding, then  $G$  is hyperbolic, each element  $g \in G$  with  $\deg(g) = 1$  is loxodromic, and there exists a  $G$ -forward invariant non-empty compact subset of  $F(G)$ .*

- (2) If there exists  $g \in G$  with  $\deg(g) \geq 2$ , if  $G$  is hyperbolic and if each element  $g \in G$  with  $\deg(g) = 1$  is loxodromic, then  $f$  is expanding.
- (3) If  $G \subset \text{Aut}(\widehat{\mathbb{C}})$  and each element  $g \in G$  is loxodromic, and if there exists a  $G$ -forward invariant non-empty compact subset of  $F(G)$ , then  $f$  is expanding.

Finally, we state the following facts about the exceptional set of an expanding rational semigroup.

**Lemma 2.9.** *Let  $G = \langle f_i : i \in I \rangle$  denote an expanding rational semigroup. Suppose that  $\text{card}(J(G)) \geq 3$ . Then we have  $E(G) \subset F(G)$ .*

*Proof.* Suppose for a contradiction that there exists  $z_0 \in E(G) \cap J(G)$ . Since  $\text{card}(J(G)) \geq 3$ , it follows from the density of the repelling fixed points (Lemma 2.3 (f)) that there exist  $z_1 \in J(G)$  and  $g_1 \in G$ , such that  $z_1 \neq z_0$ ,  $g_1(z_1) = z_1$  and  $\|g_1'(z_1)\| > 1$ . Furthermore, we have  $\text{card}(E(G)) \leq 2$  by Lemma 2.3 (c). Combining with the fact that  $g^{-1}(E(G)) \subset E(G)$  for each  $g \in G$ , we conclude that  $g_1^2(z_0) = z_0$ . Since  $G$  is expanding, we have that either  $\deg(g_1) \geq 2$  or that  $g_1$  is a loxodromic Möbius transformation by Proposition 2.8 (1). Thus, we have that  $z_0$  is an attracting fixed point of  $g_1^2$ . Let  $V$  be a neighborhood of  $z_0$  and let  $0 < c < 1$  such that  $g_1^2(V) \subset V$  and  $\|(g_1^2)'(z)\| < c$ , for each  $z \in V$ . By Lemma 2.3 (b) there exists a sequence  $(a_n)$  with  $a_n \in J(G) \setminus \{z_0\}$  such that  $\lim_n a_n = z_0$ . Then there exists a sequence  $(n_k) \in \mathbb{N}^{\mathbb{N}}$  tending to infinity and a sequence  $(b_k) \in \widehat{\mathbb{C}}^{\mathbb{N}}$  such that  $b_k \in g_1^{-2n_k}(a_{n_k})$  and  $b_k \in V$ . Hence,  $\lim_k \|(g_1^{2n_k})'(b_k)\| \leq \lim_k c^{n_k} = 0$ . Moreover, write  $g_1^2 = f_\alpha$ , for some  $m \in \mathbb{N}$  and  $\alpha \in I^m$ , and denote by  $\alpha^n := (\alpha \dots \alpha) \in I^{mn}$  the  $n$ -fold concatenation of  $\alpha$ . Let  $(\beta_k) \in I^{\mathbb{N}}$  with  $(\beta_k, a_{n_k}) \in J(\tilde{f})$ . Then  $(\alpha^{n_k} \beta_k, b_k) \in J(\tilde{f})$ . This contradicts that  $G$  is expanding and finishes the proof.  $\square$

**Lemma 2.10.** *Let  $G = \langle f_i : i \in I \rangle$  denote an expanding rational semigroup. Suppose that  $1 \leq \text{card}(J(G)) \leq 2$ . Then we have  $\text{card}(J(G)) = 1$ .*

*Proof.* Clearly, we have  $G \subset \text{Aut}(\widehat{\mathbb{C}})$  and each element of  $G$  is loxodromic by Proposition 2.8 (1). Now, suppose by way of contradiction that  $J(G) = \{a, b\}$  with  $a \neq b$ . Without loss of generality, we may assume that  $a = 0$  and  $b = \infty$ . Since  $J(G)$  is  $G$ -backward invariant, we have  $g(a) = a$  and  $g(b) = b$  for each  $g \in G$ . Thus, there exists a sequence  $(c_i) \in \mathbb{C}^I$  such that  $f_i(z) = c_i z$ , for each  $z \in \widehat{\mathbb{C}}$  and  $i \in I$ . We may assume that there exists  $i_0 \in I$  such that  $\|f_{i_0}'(a)\| > 1$ . Since  $G$  is expanding with respect to  $\{f_i : i \in I\}$ , there exists a constant  $c_0 > 1$  such that  $\|f_i'(a)\| = |c_i| \geq c_0 > 1$ , for all  $i \in I$ . Hence, we have  $\|f_i'(b)\| \leq c_0^{-1} < 1$ , for all  $i \in I$ , which gives that  $b \in F(G)$ . This contradiction proves the lemma.  $\square$

### 3. THERMODYNAMIC FORMALISM FOR EXPANDING RATIONAL SEMIGROUPS

In this section we collect some of the main results from the thermodynamic formalism in the context of expanding rational semigroups. It was shown in [Sum05] that the skew product of a finitely generated expanding rational semigroup is an open distance expanding map. We refer to [PU10] or the classical references [Bow75, Rue78, Wal82] for general results on thermodynamic formalism for expanding maps.

**3.1. Conformal measure and equilibrium states.** We first give the definition of conformal measures which are useful to investigate geometric properties of the Julia set of a rational semigroup. A general notion of conformal measure was introduced in [DU91]. For results on conformal measures in the context of expanding rational semigroups we refer to [Sum98, Sum05].



**Definition 3.1.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and let  $\tilde{\varphi} : J(\tilde{f}) \rightarrow \mathbb{R}$  be Borel measurable. A Borel probability measure  $\tilde{\nu}$  on  $J(\tilde{f})$  is called  $\tilde{\varphi}$ -conformal (for  $\tilde{f}$ ) if, for each Borel set  $A \subset J(\tilde{f})$  such that  $\tilde{f}|_A$  is injective, we have

$$\tilde{\nu}(\tilde{f}(A)) = \int_A e^{-\tilde{\varphi}} d\tilde{\nu}.$$

Next we give the fundamental definitions of topological pressure and equilibrium states.

**Definition 3.2.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and let  $\tilde{\varphi} : J(\tilde{f}) \rightarrow \mathbb{R}$  be continuous. The topological pressure  $\mathcal{P}(\tilde{\varphi}, \tilde{f})$  of  $\tilde{\varphi}$  with respect to  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is given by

$$\mathcal{P}(\tilde{\varphi}, \tilde{f}) := \sup \left\{ h(\tilde{m}) + \int \tilde{\varphi} d\tilde{m} : \tilde{m} \in \mathcal{M}_e(\tilde{f}) \right\},$$

where  $\mathcal{M}_e(\tilde{f})$  denotes the set of all  $\tilde{f}$ -invariant ergodic Borel probability measures on  $J(\tilde{f})$  and  $h(\tilde{m})$  refers to the measure-theoretic entropy of  $(\tilde{f}, \tilde{m})$ . An ergodic  $\tilde{f}$ -invariant Borel probability measure  $\tilde{\mu}$  on  $J(\tilde{f})$  is called an equilibrium state for  $\tilde{\varphi}$  if

$$\mathcal{P}(\tilde{\varphi}, \tilde{f}) = h(\tilde{\mu}) + \int \tilde{\varphi} d\tilde{\mu}.$$

The following lemma guarantees existence and uniqueness of conformal measures and equilibrium states for Hölder continuous potentials. The lemma can be proved similarly as in [Sum05, Lemma 3.6 and 3.10]. For the uniqueness of the equilibrium state, see e.g. [PU10].

For  $\omega, \kappa \in I^{\mathbb{N}}$ , we set  $d_0(\omega, \kappa) := 2^{-|\omega \wedge \kappa|}$ , where  $\omega \wedge \kappa$  denotes the longest common initial block of  $\omega$  and  $\kappa$ . The Julia set  $J(\tilde{f})$  is equipped with the metric  $\tilde{d}$  which is for  $(\omega, x), (\kappa, y) \in J(\tilde{f})$  given by  $\tilde{d}((\omega, x), (\kappa, y)) := d_0(\omega, \kappa) + d(x, y)$ , where  $d$  denotes the spherical distance on  $\widehat{\mathbb{C}}$ . We say that  $\tilde{\varphi} : J(\tilde{f}) \rightarrow \mathbb{R}$  is Hölder continuous if there exists  $\theta > 0$  such that

$$\sup \left\{ \frac{d(\tilde{\varphi}(\omega, x), \tilde{\varphi}(\kappa, y))}{\tilde{d}((\omega, x), (\kappa, y))^\theta} : (\omega, x), (\kappa, y) \in J(\tilde{f}), (\omega, x) \neq (\kappa, y) \right\} < \infty.$$

**Lemma 3.3.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\tilde{\varphi} : J(\tilde{f}) \rightarrow \mathbb{R}$  be Hölder continuous with  $\mathcal{P}(\tilde{\varphi}, \tilde{f}) = 0$ . Then we have the following.

- (1) There exists a unique  $\tilde{\varphi}$ -conformal measure  $\tilde{\nu}$  on  $J(\tilde{f})$ .
- (2) There exists a unique continuous function  $\tilde{h} : J(\tilde{f}) \rightarrow \mathbb{R}^+$  such that the probability measure  $\tilde{\mu} := \tilde{h} d\tilde{\nu}$  is  $\tilde{f}$ -invariant. Moreover, we have that  $\tilde{\mu}$  is exact (hence ergodic) and  $\tilde{\mu}$  is the unique equilibrium state for  $\tilde{\varphi}$ .

We also consider subconformal measures on  $J(G)$ .

**Definition 3.4.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  and let  $G = \langle f_i : i \in I \rangle$ . Let  $\varphi = (\varphi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be a family of measurable functions. A Borel probability measure  $\nu$  on  $J(G)$  is called  $\varphi$ -subconformal (for  $f$ ) if, for each  $i \in I$  and for each Borel set  $B \subset f_i^{-1}(J(G))$ ,

$$\nu(f_i(B)) \leq \int_B e^{-\varphi_i} d\nu.$$

Next lemma shows that the support of a subconformal measure is equal to the Julia set.

**Lemma 3.5.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Let  $\nu$  denote a  $\varphi$ -subconformal measure of a measurable family  $\varphi = (\varphi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$ . Then we have  $\text{supp}(\nu) = J(G)$ .

*Proof.* We consider two cases. If  $\text{card}(J(G)) \geq 3$  then  $E(G) \subset F(G)$  by Lemma 2.9. Then  $\text{supp}(\nu) = J(G)$  can be proved similarly as in [Sum98, Proposition 4.3]. Finally, if  $1 \leq \text{card}(J(G)) \leq 2$ , then we have  $\text{card}(J(G)) = 1$  by Lemma 2.10, which immediately gives that  $\text{supp}(\nu) = J(G)$ . The proof is complete.  $\square$

For a Borel measure  $\tilde{m}$  on  $J(\tilde{f})$  we denote by  $(\pi_{\tilde{c}})_*(\tilde{m})$  the pushforward measure, which is for each Borel set  $B \subset J(G)$  given by  $(\pi_{\tilde{c}})_*(\tilde{m})(B) := \tilde{m}(\pi_{\tilde{c}}^{-1}(B))$ . Next lemma is a straightforward generalisation of [Sum05, Lemma 3.11].

**Lemma 3.6.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Let  $\varphi = (\varphi_i)_{i \in I}$  be a Hölder family. Suppose that  $\mathcal{P}(\tilde{\varphi}, \tilde{f}) = 0$  and let  $\tilde{\nu}$  denote the unique  $\tilde{\varphi}$ -conformal measure. Then the probability measure  $\nu := (\pi_{\tilde{c}})_*(\tilde{\nu})$  is a  $\varphi$ -subconformal measure with  $\text{supp}(\nu) = J(G)$ .*

**3.2. The free energy function.** Let us now introduce the free energy function and an important family of associated measures (see [Rue78] and [Bow75, PW97, Pes97]).

**Definition 3.7.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$ . The *free energy function* for  $(f, \psi)$  is the unique function  $t : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathcal{P}(\beta\tilde{\psi} + t(\beta)\tilde{\zeta}, \tilde{f}) = 0$ , for each  $\beta \in \mathbb{R}$ . For each  $\beta \in \mathbb{R}$ , we denote by  $\tilde{\nu}_\beta$  the unique  $\beta\tilde{\psi} + t(\beta)\tilde{\zeta}$ -conformal measure for  $\tilde{f}$ , and we denote by  $\tilde{\mu}_\beta$  the unique equilibrium state for  $\beta\tilde{\psi} + t(\beta)\tilde{\zeta}$ . Moreover, we denote by  $\nu_\beta$  the pushforward measure  $(\pi_{\tilde{c}})_*(\tilde{\nu}_\beta)$  supported on  $J(G)$ . We also set

$$\alpha_0(\psi) := \frac{\int \tilde{\psi} d\tilde{\mu}_0}{\int \tilde{\zeta} d\tilde{\mu}_0}.$$

*Remark 3.8.* Using that  $f$  is expanding, one immediately verifies that, for each  $\beta \in \mathbb{R}$ , there exists a unique  $t(\beta)$  such that  $\mathcal{P}(\beta\tilde{\psi} + t(\beta)\tilde{\zeta}, \tilde{f}) = 0$ . In particular, we have that  $t(0)$  is the unique real number  $\delta$  such that  $\mathcal{P}(\delta\tilde{\zeta}, \tilde{f}) = 0$ , which is also called the critical exponent of  $f$  ([Sum05]). Since we always assume that  $f$  is non-exceptional, we have that  $t(0) = \delta > 0$ .

The following two propositions go back to work of Ruelle ([Rue78]) for shift spaces. Since the skew product map  $\tilde{f}$  is an open distance expanding map ([Sum05]), we see that  $(J(\tilde{f}), \tilde{f})$  is semi-conjugate to a shift space by choosing a Markov partition. Moreover, by [Sum00, Proposition 3.2 (f)] and Lemma 2.9,  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically exact. Then it is standard to derive the following two propositions. (See also [PW97, Pes97], where these results of Ruelle are applied to a similar kind of multifractal analysis for conformal repellers.)

**Proposition 3.9.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$ . The free energy function  $t : \mathbb{R} \rightarrow \mathbb{R}$  for  $(f, \psi)$  is convex and real analytic and its first derivative is given by*

$$t'(\beta) = -\frac{\int \tilde{\psi} d\tilde{\mu}_\beta}{\int \tilde{\zeta} d\tilde{\mu}_\beta}, \quad \text{for each } \beta \in \mathbb{R}.$$

The following proposition gives a criterion for strict convexity of the free energy function. For the readers, we give a proof.

**Proposition 3.10.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Suppose that there exists  $\gamma \in \mathbb{R}$  such that  $\mathcal{P}(\gamma\tilde{\psi}, \tilde{f}) = 0$ . Then, the following (1)-(4) are equivalent:*

- (1) *There exists  $\beta_0 \in \mathbb{R}$  such that  $t''(\beta_0) = 0$ .*

- (2)  $t'$  is constant on  $\mathbb{R}$ . In this case, we have  $t(\beta) = \delta - \beta\delta/\gamma$ .  
(3)  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ .  
(4) There exists a continuous function  $v : J(\tilde{f}) \rightarrow \mathbb{R}$  such that  $\delta\tilde{\zeta} = \gamma\tilde{\psi} + v - v \circ \tilde{f}$ .

In particular,  $t'' > 0$  on  $\mathbb{R}$  if and only if  $\tilde{\mu}_0 \neq \tilde{\mu}_\gamma$ .

Moreover, if there exists  $\beta \in \mathbb{R}$  such that  $t''(\beta) \neq 0$ , then  $t'' > 0$  on  $\mathbb{R}$  and  $t$  is strictly convex on  $\mathbb{R}$ .

*Proof.*  $(J(\tilde{f}), \tilde{f})$  is an open distance expanding map and it is topologically exact. Fix a Markov partition  $\{R_1, \dots, R_d\}$  as in [PU10, p118]. Let  $A$  be a  $d \times d$  matrix with  $a_{ij} = 0$  or 1 according to  $\tilde{f}(\text{Int}(R_i)) \cap \text{Int}(R_j)$  is empty or not. By [PU10, Theorem 4.5.7], there exists a surjective Hölder continuous map  $\pi : \Sigma \rightarrow J(\tilde{f})$  (which is almost bijective), such that  $\tilde{f} \circ \pi = \pi \circ \sigma$ , where  $\sigma : \Sigma \rightarrow \Sigma$  is the subshift of finite type constructed by  $A$ . By [PU10, Theorem 4.5.8], every Hölder continuous function  $\tilde{\varphi}$  on  $J(\tilde{f})$  defines a Hölder continuous function  $\tilde{\varphi} \circ \pi$  on  $\Sigma$ , and we have  $P(\tilde{\varphi}, \tilde{f}) = P(\tilde{\varphi} \circ \pi, \sigma)$ . Hence, the free energy function  $t : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $P(\beta\tilde{\psi} \circ \pi + t(\beta)\tilde{\zeta} \circ \pi, \sigma) = 0$ .

We first show that (1) implies (2) and (3). Suppose that there exists  $\beta_0 \in \mathbb{R}$  such that  $t''(\beta_0) = 0$ . Then it is well-known (see e.g. [MU03, p129] where the geometric potential is given by  $-\zeta$  in our notation) that there exists a Hölder continuous function  $u : \Sigma \rightarrow \mathbb{R}$  such that  $t'(\beta_0)\tilde{\zeta} \circ \pi + \tilde{\psi} \circ \pi = u - u \circ \sigma$ . It follows that  $P((-\beta t'(\beta_0) + t(\beta))\tilde{\zeta} \circ \pi, \sigma) = 0$ . Hence,  $-\beta t'(\beta_0) + t(\beta) = t(0) = \delta$ . Thus,  $t(\beta) = \delta + \beta t'(\beta_0)$ . Therefore  $t'(\beta) = t'(\beta_0)$  for all  $\beta \in \mathbb{R}$ . To determine  $t'(\beta_0)$ , note that by the assumption there exists a unique  $\gamma \in \mathbb{R}$  such that  $P(\gamma\tilde{\psi} \circ \pi, \sigma) = 0$ . Hence,  $t(\gamma) = 0$ . It follows that  $t'(\beta_0) = -\delta/\gamma$ . We have thus shown that  $t(\beta) = \delta - \beta\delta/\gamma$  and that  $\delta\tilde{\zeta} \circ \pi$  is cohomologous to  $\gamma\tilde{\psi} \circ \pi$ . Hence,  $\rho_{\delta\tilde{\zeta} \circ \pi} = \rho_{\gamma\tilde{\psi} \circ \pi}$ , where  $\rho_{\delta\tilde{\zeta} \circ \pi}$  and  $\rho_{\gamma\tilde{\psi} \circ \pi}$  are the unique equilibrium states of  $\delta\tilde{\zeta} \circ \pi$  and  $\tilde{\psi} \circ \pi$ , respectively. Since  $\pi$  defines an isomorphism of the probability spaces  $(\Sigma, \rho_{\delta\tilde{\zeta} \circ \pi})$  and  $(J(\tilde{f}), \rho_{\delta\tilde{\zeta} \circ \pi} \circ \pi^{-1})$  by [PU10, Theorem 4.5.9], we have that  $\rho_{\delta\tilde{\zeta} \circ \pi} \circ \pi^{-1}$  is the equilibrium state for  $\delta\tilde{\zeta}$ . Similarly,  $\rho_{\gamma\tilde{\psi} \circ \pi} \circ \pi^{-1}$  is the equilibrium state for  $\gamma\tilde{\psi}$ . Also,  $\rho_{\delta\tilde{\zeta} \circ \pi} \circ \pi^{-1} = \rho_{\gamma\tilde{\psi} \circ \pi} \circ \pi^{-1}$ . Since the equilibrium states of Hölder continuous potentials on  $J(\tilde{f})$  are unique ([PU10, Theorem 5.6.2]) and the probability measures  $\tilde{\mu}_0$  and  $\tilde{\mu}_\gamma$  are equilibrium states for  $\delta\tilde{\zeta}$  and  $\gamma\tilde{\zeta}$  respectively, we conclude that  $\rho_{\delta\tilde{\zeta} \circ \pi} \circ \pi^{-1} = \tilde{\mu}_0$  and  $\rho_{\gamma\tilde{\psi} \circ \pi} \circ \pi^{-1} = \tilde{\mu}_\gamma$ . Thus it follows that  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ . Hence, we have shown that (1) implies (2) and (3).

We now suppose (3). To prove (4) we proceed similarly as in the proof of [SU12, Theorem 3.1]. We consider the Perron-Frobenius operators

$$\mathcal{L}_0 : C(J(\tilde{f})) \rightarrow C(J(\tilde{f})) \quad \text{and} \quad \mathcal{L}_\gamma : C(J(\tilde{f})) \rightarrow C(J(\tilde{f})),$$

which are for  $\tilde{h} \in C(J(\tilde{f}))$  and  $(\omega, z) \in J(\tilde{f})$  given by

$$\mathcal{L}_0(\tilde{h})(\omega, z) := \sum_{(i\omega, y) \in \tilde{f}^{-1}(\omega, z)} \|\tilde{f}'_i(y)\|^{-\delta} \tilde{h}(i\omega, y) \quad \text{and} \quad \mathcal{L}_\gamma(\tilde{h})(\omega, z) := \sum_{(i\omega, y) \in \tilde{f}^{-1}(\omega, z)} e^{\gamma\psi_i(y)} \tilde{h}(i\omega, y).$$

We denote by  $\mathcal{L}_0^*$  and  $\mathcal{L}_\gamma^*$  the dual operators acting on the space  $(C(J(\tilde{f})))^*$  of bounded linear functionals on  $C(J(\tilde{f}))$ . Then it follows from [Sum05] that there exist unique continuous positive functions  $\tilde{h}_0, \tilde{h}_\gamma : J(\tilde{f}) \rightarrow \mathbb{R}^+$  such that  $\mathcal{L}_0(\tilde{h}_0) = \tilde{h}_0$  and  $\mathcal{L}_\gamma(\tilde{h}_\gamma) = \tilde{h}_\gamma$ , and unique Borel probability measures  $\tilde{\nu}_0, \tilde{\nu}_\gamma$  on  $J(\tilde{f})$  such that  $\mathcal{L}_0^*(\tilde{\nu}_0) = \tilde{\nu}_0$  and  $\mathcal{L}_\gamma^*(\tilde{\nu}_\gamma) = \tilde{\nu}_\gamma$ . Then  $\tilde{\nu}_0$  is  $\delta\tilde{\zeta}$ -conformal and  $\tilde{\nu}_\gamma$  is  $\gamma\tilde{\psi}$ -conformal (see Definition 3.1 and [DU91]). Moreover, we have that the unique equilibrium states  $\tilde{\mu}_0, \tilde{\mu}_\gamma$  are given by  $\tilde{\mu}_0 = \tilde{h}_0 d\tilde{\nu}_0$  and  $\tilde{\mu}_\gamma = \tilde{h}_\gamma d\tilde{\nu}_\gamma$  (see also Lemma 3.3). Since  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ , we have that  $\tilde{h}_0 d\tilde{\nu}_0 = \tilde{h}_\gamma d\tilde{\nu}_\gamma$ . Using this equality and conformality of  $\tilde{\nu}_0$  and  $\tilde{\nu}_\gamma$ , we obtain that, for each Borel set  $A \subset J(\tilde{f})$  such that  $\tilde{f}|_A$  is injective,

$$\tilde{\mu}_0(\tilde{f}(A)) = \int_{\tilde{f}(A)} \tilde{h}_\gamma d\tilde{\nu}_\gamma = \int_A e^{-\gamma\tilde{\psi}}(\tilde{h}_\gamma \circ \tilde{f}) d\tilde{\nu}_\gamma = \int_A e^{-\gamma\tilde{\psi}}(\tilde{h}_\gamma \circ \tilde{f}) \frac{\tilde{h}_0}{\tilde{h}_\gamma} d\tilde{\nu}_0$$

and

$$\tilde{\mu}_0(\tilde{f}(A)) = (\tilde{h}_0 d\tilde{\nu}_0)(\tilde{f}(A)) = \int_A \|\tilde{f}'\|^\delta (\tilde{h}_0 \circ \tilde{f}) d\tilde{\nu}_0.$$

We deduce that

$$(\tilde{h}_0 \circ \tilde{f}) \cdot \|\tilde{f}'\|^\delta = e^{-\gamma\tilde{\psi}} \frac{\tilde{h}_\gamma \circ \tilde{f}}{\tilde{h}_\gamma} \tilde{h}_0.$$

By taking logarithm, we have thus shown that

$$(3.1) \quad -\delta\tilde{\zeta} = -\gamma\tilde{\psi} + \log \tilde{h}_\gamma \circ \tilde{f} - \log \tilde{h}_\gamma + \log \tilde{h}_0 - \log \tilde{h}_0 \circ \tilde{f},$$

which proves (4).

We now suppose (4). Then  $P((\beta + t(\beta)\gamma/\delta)\tilde{\psi}, \tilde{f}) = 0$ . It follows that  $t(\beta) = \delta - \beta\delta/\gamma$ . Thus,  $t'$  is constant on  $\mathbb{R}$ , which proves (1) and thus finishes the proof of the proposition.  $\square$

Recall that, for a convex function  $a : \mathbb{R} \rightarrow \mathbb{R}$ , its convex conjugate  $a^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  is given by  $a^*(c) := \sup_{\beta \in \mathbb{R}} \{\beta c - a(\beta)\}$  for each  $c \in \mathbb{R}$ . We make use of the following facts about the convex conjugate. We refer to [Roc70, Theorem 23.5, 26.5] for the proofs and further details.

**Lemma 3.11.** *Let  $a : \mathbb{R} \rightarrow \mathbb{R}$  be convex and differentiable and let  $a^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  be the convex conjugate of  $a$ .*

- (1) *For each  $\beta \in \mathbb{R}$  we have that  $a^*(a'(\beta)) = a'(\beta)\beta - a(\beta)$ .*
- (2) *If  $\alpha \notin \overline{a'(\mathbb{R})}$  then  $a^*(\alpha) = \infty$ .*
- (3) *Suppose that  $a$  is twice differentiable and  $a'' > 0$  on  $\mathbb{R}$ . Then  $a'(\mathbb{R})$  is an open subset of  $\mathbb{R}$ ,  $a' : \mathbb{R} \rightarrow a'(\mathbb{R})$  is invertible,  $a^*$  is twice differentiable on  $a'(\mathbb{R})$ ,  $(a^*)'' > 0$  on  $a'(\mathbb{R})$ , and we have  $(a^*)'(a'(\beta)) = \beta$  for each  $\beta \in \mathbb{R}$ . If moreover  $a$  is real analytic, then  $a^*$  is real analytic on  $a'(\mathbb{R})$ .*

The proofs of the following two lemmata are standard (see e.g. [PW97, Pes97]). To make this article more self-contained, we include the proofs. In the next lemma we deduce analytic properties of the convex conjugate of the free energy function.

**Lemma 3.12.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $\delta > 0$  denote the critical exponent of  $f$ . Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Suppose that there exists  $\gamma \in \mathbb{R}$  such that  $\mathcal{P}(\gamma\tilde{\psi}, \tilde{f}) = 0$ . Let  $\beta \in \mathbb{R}$  and  $\alpha = -t'(\beta)$ . Then we have*

$$-t^*(-\alpha) = \beta\alpha + t(\beta) = -\frac{h(\tilde{\mu}_\beta)}{\int \tilde{\zeta} d\tilde{\mu}_\beta} > 0.$$

*If  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$  then we have  $-t^*(-\alpha_0(\psi)) = \delta$  and  $-t^*(-\alpha) = -\infty$  for each  $\alpha \neq \alpha_0(\psi)$ . If  $\tilde{\mu}_0 \neq \tilde{\mu}_\gamma$  then  $s(\alpha) := -t^*(-\alpha)$  is a strictly concave real analytic function on  $-t'(\mathbb{R})$  with maximum value  $\delta = -t^*(-\alpha_0(\psi)) > 0$ , and  $s'' < 0$  on  $-t'(\mathbb{R})$ .*

*Proof.* That  $-t^*(-\alpha) = \beta\alpha + t(\beta)$  follows from Lemma 3.11 (1). Since  $\tilde{\mu}_\beta$  is the equilibrium state for  $\beta\tilde{\psi} + t(\beta)\tilde{\zeta}$  and  $\mathcal{P}(\beta\tilde{\psi} + t(\beta)\tilde{\zeta}, \tilde{f}) = 0$ , we have  $-h(\tilde{\mu}_\beta) = \int \beta\tilde{\psi} + t(\beta)\tilde{\zeta} d\tilde{\mu}_\beta$ . Combining with Proposition 3.9, we obtain  $-t^*(-\alpha) = \beta\alpha + t(\beta) = -h(\tilde{\mu}_\beta) / \int \tilde{\zeta} d\tilde{\mu}_\beta$ .

To prove that  $-h(\tilde{\mu}_\beta) / \int \tilde{\zeta} d\tilde{\mu}_\beta > 0$  first observe that  $-h(\tilde{\mu}_\beta) / \int \tilde{\zeta} d\tilde{\mu}_\beta \geq 0$ . Now suppose for a contradiction that there exists  $\beta_0 \in \mathbb{R}$  such that  $-h(\tilde{\mu}_{\beta_0}) / \int \tilde{\zeta} d\tilde{\mu}_{\beta_0} = -t^*(t'(\beta_0)) = 0$ . We distinguish two cases according to Proposition 3.10. First suppose that  $t'$  is constant on  $\mathbb{R}$ . Then we have  $0 = -t^*(t'(\beta_0)) = -t^*(t'(0)) = t(0)$  by Lemma 3.11 (1). This gives the desired contradiction, because  $f$  is non-exceptional giving that  $t(0) = \delta > 0$ . For the remaining case, we may assume that  $t'' > 0$  on  $\mathbb{R}$ . Since  $-t^*(c) \geq 0$  for all  $c$  in the open neighbourhood  $t'(\mathbb{R})$  of  $t'(\beta_0)$  and  $-t^*(t'(\beta_0)) = 0$ , we conclude that the derivative of  $t^*$

vanishes in  $t'(\beta_0)$ . By Lemma 3.11 (3), it follows that zero is a local maximum of  $-t^*$  in a neighbourhood of  $t'(\beta_0)$ , which implies that  $-t^*$  is constant in a neighbourhood of  $t'(\beta_0)$ . However, by Lemma 3.11 (3), we have that  $(t^*)'' > 0$  on  $t'(\mathbb{R})$  which is a contradiction. We have thus shown that  $-h(\tilde{\mu}_\beta) / \int \tilde{\zeta} d\tilde{\mu}_\beta > 0$  for all  $\beta \in \mathbb{R}$ .

To verify the remaining assertions, first suppose that  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ . By Proposition 3.9 and 3.10 we then have that  $t'(\beta) = t'(0) = -\alpha_0(\psi)$  for all  $\beta \in \mathbb{R}$ . By Lemma 3.11 (1) and (2) we conclude that  $-t^*(-\alpha_0(\psi)) = -t^*(t'(0)) = t(0) = \delta$  and  $-t^*(-\alpha) = -\infty$  if  $\alpha \neq \alpha_0(\psi)$ . Now suppose that  $\tilde{\mu}_0 \neq \tilde{\mu}_\gamma$ . By Proposition 3.9 and 3.10, we have that  $t$  is strictly convex and real analytic. By Lemma 3.11 (3) we have that  $-t^*$  is a strictly concave and real analytic function on  $-t'(\mathbb{R})$ . Also,  $s'' < 0$  on  $t'(\mathbb{R})$ . Moreover, Lemma 3.11 (3) implies that the derivative of  $t^*$  vanishes in  $t'(0) = -\alpha_0(\psi)$ , which shows that  $-t^*$  attains a maximum in  $-\alpha_0(\psi)$  with  $-t^*(-\alpha_0(\psi)) = \delta$ .  $\square$

For the support of the measures  $\tilde{\nu}_\beta$  and  $\nu_\beta$  we prove the following lemma. Recall that  $\mathcal{F}(\alpha, \psi)$  is the continuous image of the Borel set  $\tilde{\mathcal{F}}(\alpha, \psi)$ . In particular,  $\mathcal{F}(\alpha, \psi)$  is a Suslin set and thus  $\nu_\beta$ -measurable. We refer to [Fed69, p65-70] for details on Suslin sets.

**Lemma 3.13.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . For each  $\beta \in \mathbb{R}$ , we have that*

$$\tilde{\nu}_\beta(\tilde{\mathcal{F}}(-t'(\beta), \psi)) = \nu_\beta(\mathcal{F}(-t'(\beta), \psi)) = 1.$$

*In particular, for each  $\alpha \in -t'(\mathbb{R})$ , we have that  $\mathcal{F}(\alpha, \psi)$  is non-empty.*

*Proof.* Let  $\beta \in \mathbb{R}$ . Since  $\tilde{\mu}_\beta$  is ergodic by Lemma 3.3, we have by Birkhoff's ergodic theorem that for  $\tilde{\mu}_\beta$ -almost every  $x \in J(\tilde{f})$ ,

$$\lim_{n \rightarrow \infty} \frac{S_n \tilde{\psi}(x)}{S_n \tilde{\zeta}(x)} = \frac{\int \tilde{\psi} d\tilde{\mu}_\beta}{\int \tilde{\zeta} d\tilde{\mu}_\beta}.$$

Since  $\int \tilde{\psi} d\tilde{\mu}_\beta / \int \tilde{\zeta} d\tilde{\mu}_\beta = -t'(\beta)$  by Proposition 3.9 and  $\tilde{\mu}_\beta$  and  $\tilde{\nu}_\beta$  are equivalent by Lemma 3.3, we have that  $\tilde{\nu}_\beta(\tilde{\mathcal{F}}(-t'(\beta), \psi)) = 1$ . Consequently, we have that  $\nu_\beta(\mathcal{F}(-t'(\beta), \psi)) = 1$ .  $\square$

*Remark 3.14.* Under the assumptions of Lemma 3.13 there exists a Borel measurable subset  $A \subset \mathcal{F}(-t'(\beta), \psi)$  such that  $\nu_\beta(A) = 1$ .

*Proof.* Since the Borel measure  $\tilde{\nu}_\beta$  is regular, there exists a family of compact subsets  $(K_n)_{n \in \mathbb{N}}$ ,  $K_n \subset \tilde{\mathcal{F}}(-t'(\beta), \psi)$ , such that  $\tilde{\nu}_\beta(\tilde{\mathcal{F}}(-t'(\beta), \psi) \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0$ . Hence, for the Borel set  $\bigcup_{n \in \mathbb{N}} \pi_{\hat{\mathbb{C}}}(K_n) \subset \mathcal{F}(-t'(\beta), \psi)$ , we have that  $\nu_\beta(\bigcup_{n \in \mathbb{N}} \pi_{\hat{\mathbb{C}}}(K_n)) = 1$ .  $\square$

#### 4. MULTIFRACTAL FORMALISM

To establish the multifractal formalism for expanding multi-maps, we investigate the local dimension of the measures  $\nu_\beta$  for  $\beta \in \mathbb{R}$  (see Definition 3.7). To state the next lemma, we have to make a further definition. Recall that  $\alpha_0(\psi) := \int \tilde{\psi} d\tilde{\mu}_0 / \int \tilde{\zeta} d\tilde{\mu}_0$  for a Hölder family  $\psi$ .

**Definition 4.1.** Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$ . For  $\alpha \in \mathbb{R}$  we define

$$\mathcal{F}^\sharp(\alpha, \psi) := \begin{cases} \left\{ x \in J(\tilde{f}) : \limsup_{n \rightarrow \infty} S_n \tilde{\psi}(x) / S_n \tilde{\zeta}(x) \geq \alpha \right\}, & \text{for } \alpha \geq \alpha_0(\psi), \\ \left\{ x \in J(\tilde{f}) : \liminf_{n \rightarrow \infty} S_n \tilde{\psi}(x) / S_n \tilde{\zeta}(x) \leq \alpha \right\}, & \text{for } \alpha < \alpha_0(\psi). \end{cases}$$

Moreover, we set  $\mathcal{F}^\sharp(\alpha, \psi) := \pi_{\hat{\mathbb{C}}}(\mathcal{F}^\sharp(\alpha, \psi))$ .

The proof of the following lemma mimics the proof of [Sum98, Theorem 3.4], where  $v_0$  is considered. To state the lemma, let  $B(z, r)$  denote the spherical ball of radius  $r$  centred at  $z \in \widehat{\mathbb{C}}$ . Recall from Lemma 3.5 that  $v_\beta(B(z, r)) > 0$  for each  $z \in J(G)$  and  $r > 0$ . Thus, we have that  $\log v_\beta(B(z, r))$  is a well-defined real number.

**Lemma 4.2.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Let  $\alpha, \beta \in \mathbb{R}$  such that, either (1)  $\alpha \geq \alpha_0(\psi)$  and  $\beta \leq 0$ , or (2)  $\alpha \leq \alpha_0(\psi)$  and  $\beta \geq 0$ . For each  $z \in \mathcal{F}^\sharp(\alpha, \psi)$  we then have that*

$$0 \leq \liminf_{r \rightarrow 0} \frac{\log v_\beta(B(z, r))}{\log r} \leq t(\beta) + \beta\alpha.$$

*Proof.* We only consider the case that  $\alpha \geq \alpha_0(\psi)$  and  $\beta \leq 0$ . The remaining case can be proved in a similar fashion. Let  $z \in \mathcal{F}^\sharp(\alpha, \psi)$ . There exists  $\omega \in I^\mathbb{N}$  such that  $(\omega, z) \in \tilde{\mathcal{F}}^\sharp(\alpha, \psi)$ . Since  $\alpha \geq \alpha_0(\psi)$  we have

$$(4.1) \quad \limsup_{n \rightarrow \infty} \frac{S_n \tilde{\psi}((\omega, z))}{S_n \tilde{\zeta}((\omega, z))} \geq \alpha.$$

Since  $(\omega, z) \in J(\tilde{f})$  we have  $z \in J_\omega$  by Lemma 2.6. We set  $z_n := f_{\omega_n}(z)$  for each  $n \in \mathbb{N}$ . By Proposition 2.1 we have  $z_n \in J(G)$ . Since  $f$  is expanding, we have that  $G$  is hyperbolic and there exists a  $G$ -forward invariant non-empty compact subset of  $F(G)$  by Proposition 2.8 (1). Hence, there exists  $R > 0$  such that, for each  $n \in \mathbb{N}$ , there exists a holomorphic branch  $\phi_n : B(z_n, R) \rightarrow \widehat{\mathbb{C}}$  of  $f_{\omega_n}^{-1}$  such that  $f_{\omega_n} \circ \phi_n = \text{id}|_{B(z_n, R)}$  and  $\phi(f_{\omega_n}(z)) = z$ . By Koebe's distortion theorem, there exist constants  $c_1 > 0$  and  $c_2 > 1$  such that for each  $n \in \mathbb{N}$ ,

$$(4.2) \quad \phi_n(B(z_n, c_2^{-1}R)) \subset B(z, c_1^{-1}R \|\phi'_n(z_n)\|).$$

Using that  $v_\beta$  is  $\beta\psi + t(\beta)\zeta$ -subconformal by Lemma 3.6 and the set inclusion in (4.2), we obtain that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} v_\beta(f_{\omega_n}(\phi_n(B(z_n, c_2^{-1}R)))) &\leq \int_{\phi_n(B(z_n, c_2^{-1}R))} e^{-S_n(\beta\tilde{\psi}+t(\beta)\tilde{\zeta})(\omega, \cdot)} d\nu_\beta \\ &\leq v_\beta(B(z, c_1^{-1}R \|\phi'_n(z_n)\|)) \max_{y \in \phi_n(B(z_n, c_2^{-1}R))} e^{-S_n(\beta\tilde{\psi}+t(\beta)\tilde{\zeta})(\omega, y)}. \end{aligned}$$

Since  $\text{supp}(v_\beta) = J(G)$  by Lemma 3.5 and by the compactness of  $J(G)$ , there exists a constant  $M > 0$  such that  $v_\beta(f_{\omega_n}(\phi_n(B(z_n, c_2^{-1}R)))) > M$  for all  $n \in \mathbb{N}$ . Using that  $f$  is expanding and that  $\beta\tilde{\psi} + t(\beta)\tilde{\zeta}$  is Hölder continuous, one verifies that there exists a constant  $C > 1$  such that, for all  $n \in \mathbb{N}$  and for all  $x \in \phi_n(B(z_n, c_2^{-1}R))$ ,

$$\left| S_n(\beta\tilde{\psi} + t(\beta)\tilde{\zeta})(\omega, x) - S_n(\beta\tilde{\psi} + t(\beta)\tilde{\zeta})(\omega, z) \right| \leq \log C.$$

From  $\beta \leq 0$  and (4.1) it follows that there exists a sequence  $(n_j) \in \mathbb{N}^\mathbb{N}$  tending to infinity, such that, for each  $\varepsilon > 0$ , we have for all  $j$  sufficiently large,

$$\max_{x \in \phi_{n_j}(B(z_{n_j}, c_2^{-1}R))} e^{-S_{n_j}(\beta\tilde{\psi}+t(\beta)\tilde{\zeta})(\omega, x)} \leq C e^{-S_{n_j}\tilde{\zeta}(\omega, z)(\beta\alpha+t(\beta)-\beta\varepsilon)} = C \|\phi'_{n_j}(z_{n_j})\|^{-(\beta\alpha+t(\beta)-\beta\varepsilon)}.$$

We have thus shown that  $0 < M \leq C v_\beta(B(z, c_1^{-1}R \|\phi'_{n_j}(z_{n_j})\|)) \|\phi'_{n_j}(z_{n_j})\|^{-(\beta\alpha+t(\beta)-\beta\varepsilon)}$  for all  $j$  sufficiently large. Set  $r_j := c_1^{-1}R \|\phi'_{n_j}(z_{n_j})\|$ . Clearly, we have that  $\lim_j r_j = 0$ . Hence, we have

$$0 \leq \liminf_{r \rightarrow 0} \frac{\log v_\beta(B(z, r))}{\log r} \leq \beta\alpha + t(\beta) - \beta\varepsilon.$$

Since  $\varepsilon$  was arbitrary, the proof is complete.  $\square$

**Lemma 4.3.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Then we have the following.*

(1) *For each  $\alpha \in \mathbb{R}$  we have*

$$-t^*(-\alpha) = \begin{cases} \inf_{\beta \leq 0} \{t(\beta) + \beta\alpha\}, & \text{for } \alpha \geq \alpha_0(\psi), \\ \inf_{\beta \geq 0} \{t(\beta) + \beta\alpha\}, & \text{for } \alpha \leq \alpha_0(\psi). \end{cases}$$

(2) *Let  $\alpha \in \mathbb{R}$ . If  $-t^*(-\alpha) < 0$  then  $\mathcal{F}(\alpha, \psi) = \mathcal{F}^\sharp(\alpha, \psi) = \emptyset$ . In particular, we have that  $\mathcal{F}(\alpha, \psi) = \mathcal{F}^\sharp(\alpha, \psi) = \emptyset$ , if  $\alpha \notin \overline{-t'(\mathbb{R})}$ .*

*Proof.* To prove (1), first recall that  $\alpha_0(\psi) = -t'(0)$  by Proposition 3.9. Hence, we have that  $-t^*(-\alpha_0(\psi)) = -t^*(t'(0)) = t(0)$  by Lemma 3.11 (1). Now we only consider the case that  $\alpha \geq \alpha_0(\psi)$ . The remaining case can be proved similarly. If  $\beta > 0$  then we have that  $t(\beta) + \beta\alpha \geq t(\beta) + \beta\alpha_0(\psi) \geq -t^*(-\alpha_0(\psi)) = t(0)$ . Hence, we have  $-t^*(-\alpha) = \inf_{\beta \leq 0} \{t(\beta) + \beta\alpha\}$ .

For the proof of (2), suppose for a contradiction that there exists  $\alpha \in \mathbb{R}$  and  $z \in \mathcal{F}^\sharp(\alpha, \psi)$  such that  $-t^*(-\alpha) < 0$ . Again, we only consider the case that  $\alpha \geq \alpha_0(\psi)$ . By (1) there exists  $\beta \leq 0$  such that  $t(\beta) + \beta\alpha < 0$ . This contradicts Lemma 4.2 and thus proves the first assertion in (2). Finally, if  $\alpha \notin \overline{-t'(\mathbb{R})}$  then we have  $-t^*(-\alpha) = -\infty$  by Lemma 3.11 (2). Hence, we have  $\mathcal{F}^\sharp(\alpha, \psi) = \emptyset$ . Since  $\mathcal{F}(\alpha, \psi) \subset \mathcal{F}^\sharp(\alpha, \psi)$ , the proof is complete.  $\square$

For an expanding multi-map which satisfies the open set condition, we prove the following lower bound for the Hausdorff dimension of  $\nu_\beta$  by using estimates from [Sum05, Section 5]. Related results and similar arguments can be found in [PW97, Lemma 2] for conformal repellers, and in [MU03, Theorem 4.4.2] for graph directed Markov systems. Recall that, for a Borel probability measure  $\nu$  on  $J(G)$ , the Hausdorff dimension of  $\nu$  (cf. [Fal03]) is given by

$$\dim_H(\nu) := \inf \{ \dim_H(A) : A \subset J(G) \text{ is a Borel set with } \nu(A) = 1 \}.$$

**Proposition 4.4.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $f$  satisfies the open set condition. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . For each  $\beta \in \mathbb{R}$  we have that*

$$\dim_H(\nu_\beta) \geq -t^*(t'(\beta)).$$

*In particular, we have  $\dim_H(\mathcal{F}(\alpha, \psi)) \geq -t^*(-\alpha)$  for each  $\alpha \in \overline{-t'(\mathbb{R})}$ .*

*Proof.* We use some estimates and notations from [Sum05, Section 5]. Suppose that  $f$  satisfies the open set condition with open set  $U \subset \widehat{\mathbb{C}}$ . We may assume that there exists  $\varepsilon > 0$  such that  $B(\overline{U}, 2\varepsilon) \cap P(G) = \emptyset$ . Let  $\overline{U} = \bigcup_{j=1}^k K_j$  be a measurable partition such that  $\text{Int}(K_j) \neq \emptyset$  and  $\text{diam}(K_j) \leq \varepsilon/10$  for each  $j \in \{1, \dots, k\}$ .

Let  $\beta \in \mathbb{R}$  and  $\alpha = -t'(\beta)$ . Our main task is to prove that there exists a constant  $C > 0$  with the property that, for each  $\Delta > 0$  there exist  $r_0(\Delta) > 0$  and a Borel set  $\tilde{E}(\alpha, \Delta) \subset J(\tilde{f})$  with  $\tilde{\nu}_\beta(\tilde{E}(\alpha, \Delta)) > 0$ , such that for all  $r \leq r_0(\Delta)$  and  $z \in J(G)$  we have

$$(4.3) \quad \tilde{\nu}_\beta \left( \pi_{\widehat{\mathbb{C}}}^{-1}(B(z, r)) \cap \tilde{E}(\alpha, \Delta) \right) \leq Cr^{t(\beta) + \beta\alpha - \Delta}.$$

For the Borel probability measure  $\nu_{\beta,\Delta}$  on  $J(G)$ , given by  $\nu_{\beta,\Delta}(A) := \tilde{\nu}_\beta(\pi_{\mathbb{C}}^{-1}(A) \cap \tilde{E}(\alpha, \Delta)) / \tilde{\nu}_\beta(\tilde{E}(\alpha, \Delta))$ , for  $A \subset J(G)$ , we then have for each  $z \in J(G)$ ,

$$\liminf_{r \rightarrow 0} \frac{\log \nu_{\beta,\Delta}(B(z, r))}{\log r} = \liminf_{r \rightarrow 0} \frac{\log \tilde{\nu}_\beta(\pi_{\mathbb{C}}^{-1}(B(z, r)) \cap \tilde{E}(\alpha, \Delta))}{\log r} \geq t(\beta) + \beta\alpha - \Delta.$$

Hence, we have  $\dim_H(\nu_{\beta,\Delta}) \geq t(\beta) + \beta\alpha - \Delta$  by [You82], which gives  $\dim_H(\nu_\beta) \geq \dim_H(\nu_{\beta,\Delta}) \geq t(\beta) + \beta\alpha - \Delta$ . Letting  $\Delta$  tend to zero, gives that  $\dim_H(\nu_\beta) \geq t(\beta) + \beta\alpha$ . Finally, since there exists a Borel subset  $A$  of  $J(G)$  with  $A \subset \mathcal{F}(\alpha, \psi)$  and  $\nu_\beta(A) = 1$  by Lemma 3.13 and Remark 3.14, we have  $\dim_H(\mathcal{F}(\alpha, \psi)) \geq \dim_H(\nu_\beta) \geq t(\beta) + \beta\alpha$ , which finishes the proof.

To prove (4.3), let  $z \in J(G)$  and let  $\Delta > 0$ . By Lemma 3.13 we have  $\tilde{\nu}_\beta(\tilde{\mathcal{F}}(\alpha, \psi)) = 1$ . By Egoroff's Theorem, there exist a Borel set  $\tilde{E}(\alpha, \Delta) \subset \tilde{\mathcal{F}}(\alpha, \psi)$  with  $\tilde{\nu}_\beta(\tilde{E}(\alpha, \Delta)) > 0$  and  $N(\Delta) \in \mathbb{N}$  such that

$$(4.4) \quad \inf_{y \in \tilde{E}(\alpha, \Delta)} \beta \frac{S_n \tilde{\Psi}(y)}{S_n \tilde{\zeta}(y)} \geq \beta\alpha - \Delta, \text{ for all } n \geq N(\Delta).$$

With the notation from [Sum05, Lemma 5.15], we have for each  $r > 0$ ,

$$(4.5) \quad \pi_{\mathbb{C}}^{-1}(B(z, r)) \cap J(\tilde{f}) \subset \bigcup_{i=1}^p \eta^i(\pi_{\mathbb{C}}^{-1}(B(K_{v_i}, \varepsilon/5)) \cap J(\tilde{f})),$$

where  $p \in \mathbb{N}$ ,  $1 \leq v_i \leq k$  and  $\eta^i(\kappa, y) = (\omega^i \kappa, \gamma_1^i \dots \gamma_{l_i}^i(y))$ ,  $\kappa \in I^{\mathbb{N}}$ ,  $y \in B(K_{v_i}, \varepsilon/5)$ , for each  $1 \leq i \leq p$ . Here, we have  $\omega^i \in I^{l_i}$ ,  $l_i \in \mathbb{N}$  and  $\gamma_1^i \dots \gamma_{l_i}^i$  is an inverse branch of  $f_{\omega^i}$  in a neighborhood of  $B(K_{v_i}, \varepsilon/5)$  and  $\gamma_j^i$  is an inverse branch of  $f_{\omega_j^i}$ , for every  $1 \leq i \leq p$ . It is important to note that  $p \leq C_4$ , for some constant  $C_4$  independent of  $r$  and  $z$  by [Sum05, (12)]. Moreover, by [Sum05, (13)], we have that

$$(4.6) \quad \|(\gamma_1^i \circ \dots \circ \gamma_{l_i}^i)'(y)\| \leq C_5 r, \quad y \in B(K_{v_i}, \varepsilon/5),$$

with some constant  $C_5$  independent of  $r$  and  $y$ . In particular, we have that  $l_i$  tends to infinity as  $r$  tends to zero. Hence, there exists  $r_0(\Delta) > 0$  such that, for each  $z$  and  $r \leq r_0(\Delta)$ , we have  $l_i \geq N(\Delta)$  for  $1 \leq i \leq p$  in (4.5). Then we obtain by (4.5) and  $\beta\tilde{\Psi} + t(\beta)\tilde{\zeta}$ -conformality of  $\tilde{\nu}_\beta$  that

$$(4.7) \quad \begin{aligned} \tilde{\nu}_\beta(\pi_{\mathbb{C}}^{-1}(B(z, r)) \cap \tilde{E}(\alpha, \Delta)) &\leq \sum_{i=1}^p \tilde{\nu}_\beta(\eta^i(\pi_{\mathbb{C}}^{-1}(B(K_{v_i}, \varepsilon/5))) \cap \tilde{E}(\alpha, \Delta)) \\ &= \sum_{i=1}^p \int_{\pi_{\mathbb{C}}^{-1}(B(K_{v_i}, \varepsilon/5))} e^{S_{l_i}(\beta\tilde{\Psi} + t(\beta)\tilde{\zeta})(\eta^i)} 1_{\tilde{E}(\alpha, \Delta)}(\eta^i) d\tilde{\nu}_\beta. \end{aligned}$$

The estimate in (4.4) gives that

$$(4.8) \quad 1_{\tilde{E}(\alpha, \Delta)} S_{l_i}(\beta\tilde{\Psi} + t(\beta)\tilde{\zeta}) = 1_{\tilde{E}(\alpha, \Delta)} S_{l_i} \tilde{\zeta}(t(\beta) + \beta S_{l_i} \tilde{\Psi} / S_{l_i} \tilde{\zeta}) \leq 1_{\tilde{E}(\alpha, \Delta)} S_{l_i} \tilde{\zeta}(t(\beta) + \beta\alpha - \Delta).$$

Combining the estimates in (4.6), (4.7) and (4.8), we obtain

$$\tilde{\nu}_\beta(\pi_{\mathbb{C}}^{-1}(B(z, r)) \cap \tilde{E}(\alpha, \Delta)) \leq C_4 (C_5 r)^{t(\beta) + \beta\alpha - \Delta},$$

which completes the proof.  $\square$

We can now state the main result of this section, which establishes the multifractal formalism for Hölder families associated with expanding rational semigroups.

**Theorem 4.5.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding. Let  $\psi = (\psi_i)_{i \in I}$  be a Hölder family associated with  $f$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Suppose there exists  $\gamma \in \mathbb{R}$  such that  $\mathcal{P}(\gamma\tilde{\zeta}, \tilde{f}) = 0$ . Let  $\alpha_\pm := \alpha_\pm(\psi)$  and  $\alpha_0 := \alpha_0(\psi)$ . Then we have the following.*



(1) If  $\alpha_- = \alpha_+$  then we have that  $\alpha_- = \alpha_0 = \alpha_+$ ,  $-t'(\mathbb{R}) = \{\alpha_0\}$  and  $\mathcal{F}(\alpha, \psi)$  is non-empty if and only if  $\alpha = \alpha_0$ . If  $\alpha_- < \alpha_+$  then we have that  $-t'(\mathbb{R}) = (\alpha_-, \alpha_+)$ , each  $\mathcal{F}(\alpha, \psi)$  is non-empty for  $\alpha \in (\alpha_-, \alpha_+)$ ,  $s(\alpha) := -t^*(-\alpha)$  is a strictly concave real analytic positive function on  $(\alpha_-, \alpha_+)$  with maximum value  $\delta = -t^*(-\alpha_0) > 0$ , and  $s'' < 0$  on  $(\alpha_-, \alpha_+)$ .

(2) For each  $\alpha \in \mathbb{R}$  we have that

$$\dim_H(\mathcal{F}(\alpha, \psi)) \leq \dim_H(\mathcal{F}^\sharp(\alpha, \psi)) \leq \max\{-t^*(-\alpha), 0\}.$$

(3) If  $f$  satisfies the open set condition, then for each  $\alpha \in -t'(\mathbb{R})$  we have that

$$\dim_H(\mathcal{F}(\alpha, \psi)) = \dim_H(\mathcal{F}^\sharp(\alpha, \psi)) = -t^*(-\alpha) > 0.$$

In particular, we have  $\dim_H(\mathcal{F}(\alpha_0, \psi)) = \delta > 0$ .

*Proof.* We start with the proof of (1). We distinguish two cases. First suppose that  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ . Then we have that  $-t'(\mathbb{R}) = \{-t'(0)\} = \{\alpha_0\}$  by Propositions 3.10 and 3.9. By Lemma 3.13 we have that  $\mathcal{F}(\alpha_0, \psi) \neq \emptyset$ . By Lemma 4.3 (2), we have that  $\mathcal{F}(\alpha, \psi) = \emptyset$  if  $\alpha \neq \alpha_0$ . Hence, we have that  $\alpha_- = \alpha_0 = \alpha_+$ . Now suppose that  $\tilde{\mu}_0 \neq \tilde{\mu}_\gamma$ . Then  $t'' > 0$  on  $\mathbb{R}$  by Proposition 3.10 and we have  $\mathcal{F}(\alpha, \psi) \neq \emptyset$  for  $\alpha \in -t'(\mathbb{R})$  by Lemma 3.13. Combining with Lemma 4.3 (2), we obtain that  $-t'(\mathbb{R}) = (\alpha_-, \alpha_+)$ . That  $s(\alpha) := -t^*(-\alpha)$  is a strictly concave real analytic positive function on  $(\alpha_-, \alpha_+)$  with maximum value  $\delta = -t^*(-\alpha_0) > 0$  and  $s'' < 0$  on  $(\alpha_-, \alpha_+)$  follows from Lemma 3.12.

To prove (2), let  $\alpha \in \mathbb{R}$  and suppose that  $\alpha \geq \alpha_0$ . The case  $\alpha < \alpha_0$  can be proved similarly. Since the upper bound in (2) clearly holds if  $\mathcal{F}^\sharp(\alpha, \psi) = \emptyset$  we may assume that  $\mathcal{F}^\sharp(\alpha, \psi) \neq \emptyset$ . Hence, we have  $-t^*(-\alpha) \geq 0$  by Lemma 4.3 (2). Let  $z \in \mathcal{F}^\sharp(\alpha, \psi)$ . By Lemma 4.2 we have for each  $\beta \leq 0$ ,

$$(4.9) \quad 0 \leq \liminf_{r \rightarrow 0} \frac{\log v_\beta(B(z, r))}{\log r} \leq t(\beta) + \beta\alpha.$$

Since  $\inf_{\beta \leq 0} \{t(\beta) + \beta\alpha\} = -t^*(-\alpha)$  by Lemma 4.3 (1), it follows from (4.9) and [Fal03, Proposition 4.9 (b)] and its proof that we have  $\dim_H(\mathcal{F}^\sharp(\alpha, \psi)) \leq -t^*(-\alpha)$ . Since  $\mathcal{F}(\alpha, \psi) \subset \mathcal{F}^\sharp(\alpha, \psi)$ , the proof of (2) is complete.

To prove (3), suppose that  $f$  satisfies the open set condition and let  $\alpha = -t'(\beta)$  for some  $\beta \in \mathbb{R}$ . By Proposition 4.4 we have  $\dim_H(\mathcal{F}(\alpha, \psi)) \geq -t^*(-\alpha)$ . By Lemma 3.12 we have  $-t^*(-\alpha) > 0$ . Combining these estimates with the upper bound in (2) completes the proof of the theorem.  $\square$

The next lemma shows that, for a Bernoulli family  $\psi$ , a trivial multifractal spectrum occurs in a very special situation. For a compact metric space  $X$ , we denote by  $C(X)$  the space of all complex-valued continuous functions endowed with the supremum norm.

**Proposition 4.6.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $\deg(f_{i_0}) \geq 2$  for some  $i_0 \in I$ . Let  $(c_i)_{i \in I}$  be a family of negative numbers and let  $\psi = (\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be given by  $\psi_i(z) = c_i$  for each  $i \in I$  and  $z \in f_i^{-1}(J(G))$ . Let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Let  $\gamma$  be the unique number such that  $\mathcal{P}(\gamma\tilde{\psi}, \tilde{f}) = 0$ . Then we have  $\alpha_-(\psi) = \alpha_+(\psi)$  if and only if there exist an automorphism  $\varphi \in \text{Aut}(\hat{\mathbb{C}})$ , complex numbers  $(a_i)_{i \in I}$  and  $\lambda \in \mathbb{R}$  such that for all  $i \in I$ ,*

$$(4.10) \quad \varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)} \quad \text{and} \quad \log \deg(f_i) = \lambda c_i.$$

Moreover, if the assertions in (4.10) hold, then we have  $\lambda = -(\gamma/\delta)$ .

*Proof.* First note that we have  $\alpha_-(\psi) = \alpha_+(\psi)$  if and only if  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$  by Theorem 4.5 (1) and Proposition 3.10. Now suppose that  $\alpha_-(\psi) = \alpha_+(\psi)$ . Then  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ . By Proposition 3.10 there exists a continuous function  $v : J(\tilde{f}) \rightarrow \mathbb{R}$  such that  $\delta\tilde{\zeta} = \gamma\tilde{\psi} + v - v \circ \tilde{f}$ .

For each  $n \in \mathbb{N}$ ,  $\xi \in I^n$  and  $u \in \mathbb{R}$ , we denote by  $p(u, \xi)$  the topological pressure of the potential  $u \log \|f'_\xi\| : J_{\bar{\xi}} \rightarrow \mathbb{R}$  with respect to  $f_{\bar{\xi}}$ , where  $\bar{\xi} := (\xi_1, \dots, \xi_n, \xi_1, \dots, \xi_n, \dots) \in I^\mathbb{N}$ . Note that  $J_{\bar{\xi}} = J(f_{\bar{\xi}})$ . Our next aim is to show that the function  $u \mapsto p(u, \xi)$ ,  $u \in \mathbb{R}$ , is constant. By [PU10, Theorem 5.6.5] we have that, for each  $u \in \mathbb{R}$  there exists an  $f_{\bar{\xi}}$ -invariant Borel probability measure  $m$  on  $J_{\bar{\xi}}$  such that

$$(4.11) \quad \frac{\partial}{\partial u} p(u, \xi) = \int_{J_{\bar{\xi}}} \log \|f'_\xi\| dm.$$

Denote by  $\tilde{m}$  the Borel probability measure supported on  $J_{\bar{\xi}}$  which is given by  $\tilde{m}(\{\bar{\xi}\} \times A) := m(A)$ , for each Borel set  $A \subset J_{\bar{\xi}}$ . Then  $\tilde{m}$  is  $\tilde{f}^n|_{J_{\bar{\xi}}}$ -invariant. From this and (3.1) we deduce that

$$(4.12) \quad \frac{\partial}{\partial u} p(u, \xi) = - \int_{J_{\bar{\xi}}} S_n \tilde{\zeta} d\tilde{m} = -(\gamma/\delta) \int_{J_{\bar{\xi}}} S_n \tilde{\psi} d\tilde{m} = -(\gamma/\delta) \sum_{i=1}^n c_{\xi_i}.$$

Hence, the function  $u \mapsto p(u, \xi)$ ,  $u \in \mathbb{R}$ , is constant. Now, similarly as in [SU12, Proof of Theorem 3.1], using Zdunik's theorem ([Zdu90]), we obtain that there exist an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$  and complex numbers  $(a_i)_{i \in I}$  such that

$$\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)}, \quad z \in \widehat{\mathbb{C}}.$$

Since  $\deg(f_{i_0}) \geq 2$  and  $f$  is expanding, it follows that  $\deg(f_i) \geq 2$  for all  $i \in I$ . Moreover, by combining (4.11) and (4.12), we have that  $\log \deg(f_i) = -(\gamma/\delta) c_i$  for each  $i \in I$ .

To prove the converse implication, suppose that there exist an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$ , complex numbers  $(a_i)_{i \in I}$  and  $\lambda \in \mathbb{R}$ , such that  $\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)}$  and  $\log \deg(f_i) = \lambda c_i$ , for each  $i \in I$ . It follows that, for each  $n \in \mathbb{N}$ ,  $\xi \in I^n$  and  $z \in J_{\bar{\xi}}$  such that  $f_{\bar{\xi}}(z) = z$ , we have that

$$\log \|f'_\xi(z)\| = \sum_{i=1}^n \log \deg(f_{\xi_i}) = \lambda \sum_{i=1}^n c_{\xi_i}.$$

Hence, we have  $-S_n \tilde{\zeta}(\bar{\xi}, z) = \lambda S_n \tilde{\psi}(\bar{\xi}, z)$ . Since  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is an open, distance expanding, topologically transitive map (see e.g. Lemma 2.3 (d) for the transitivity), it follows from a Livsic type theorem that there exists a continuous function  $\tilde{h} : J(\tilde{f}) \rightarrow \mathbb{R}$  such that  $-\tilde{\zeta} = \lambda \tilde{\psi} + \tilde{h} - \tilde{h} \circ \tilde{f}$  (see e.g. [PU10, Proposition 4.4.5]). In particular, we have  $-\delta\tilde{\zeta} = \delta\lambda\tilde{\psi} + \delta(\tilde{h} - \tilde{h} \circ \tilde{f})$ , which shows that  $\lambda = -(\gamma/\delta)$ . Thus, the potentials  $\delta\tilde{\zeta}$  and  $\gamma\tilde{\psi}$  have the same equilibrium state, which means that  $\tilde{\mu}_0 = \tilde{\mu}_\gamma$ . The proof is complete.  $\square$

## 5. APPLICATION TO RANDOM COMPLEX DYNAMICS

The first lemma relates the Hölder exponent of a function to  $Q_*$  (see Definition 1.1).

**Lemma 5.1.** *Let  $U \subset \widehat{\mathbb{C}}$  be an open set and let  $\rho : U \rightarrow \mathbb{C}$  be a bounded function. Then we have for each  $z \in U$ ,*

$$\text{Höl}(\rho, z) = Q_*(\rho, z).$$

*Proof.* Let  $\beta > \text{Höl}(\rho, z)$ . Then we have  $\limsup_{y \rightarrow z, y \neq z} |\rho(y) - \rho(z)|/d(y, z)^\beta = \infty$ , which implies that  $\limsup_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| - \beta \log d(y, z) = \infty$ . Hence, we have that

$$\limsup_{y \rightarrow z, y \neq z} (-\log d(y, z)) \left( \frac{\log |\rho(y) - \rho(z)|}{-\log d(y, z)} + \beta \right) = \infty.$$

Since  $\lim_{y \rightarrow z, y \neq z} (-\log d(y, z)) = \infty$ , we conclude that  $\limsup_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| / (-\log d(y, z)) + \beta \geq 0$ , which implies  $\liminf_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| / \log d(y, z) \leq \beta$ . We have thus shown that  $Q_*(\rho, z) \leq \text{Höl}(\rho, z)$ .

Let  $\beta < \text{Höl}(\rho, z)$ . Then we have  $\lim_{y \rightarrow z, y \neq z} |\rho(y) - \rho(z)| / d(y, z)^\beta = 0$ , which implies that  $\lim_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| - \beta \log d(y, z) = -\infty$ . Hence, we have that

$$\lim_{y \rightarrow z, y \neq z} (-\log d(y, z)) \left( \frac{\log |\rho(y) - \rho(z)|}{-\log d(y, z)} + \beta \right) = -\infty.$$

We conclude that  $\limsup_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| / (-\log d(y, z)) + \beta \leq 0$ , which then implies that  $\liminf_{y \rightarrow z, y \neq z} \log |\rho(y) - \rho(z)| / \log d(y, z) \geq \beta$ . We have thus shown that  $Q_*(\rho, z) \geq \text{Höl}(\rho, z)$  and the proof of the lemma is complete.  $\square$

The following lemma allows us to investigate the Hölder exponent of a non-constant unitary eigenfunction of  $M_\tau$  by means of ergodic sums with respect to the skew product associated with a rational semigroup.

**Lemma 5.2.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $f$  satisfies the separation condition. Let  $(p_i)_{i \in I} \in (0, 1)^I$  be a probability vector, let  $\tau := \sum_{i \in I} p_i \delta_{f_i}$ , and let  $\rho \in C(\widehat{\mathbb{C}})$  be a non-constant function belonging to  $U_\tau$ . Let  $\psi = (\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be given by  $\psi_i(z) := \log p_i$  for each  $i \in I$ . Then for each  $(\omega, z) \in J(\tilde{f})$  we have that*

$$\liminf_{n \rightarrow \infty} \frac{S_n \tilde{\psi}((\omega, z))}{S_n \tilde{\zeta}((\omega, z))} = Q_*(\rho, z) \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{S_n \tilde{\psi}((\omega, z))}{S_n \tilde{\zeta}((\omega, z))} = Q^*(\rho, z).$$

*Proof.* We proceed similarly as in the proof of [Sum11, Lemma 5.48]. By [Sum11, Theorem 3.15 (10)] we may assume that  $M_\tau(\rho) = \rho$ . Since  $f$  satisfies the separation condition, we conclude that there exists  $r_0 > 0$  such that, for all  $i, j \in I$  with  $i \neq j$  and  $y \in f_i^{-1}(J(G))$ , we have  $f_j(B(y, r_0)) \subset F(G)$ .

Let  $(\omega, z) \in J(\tilde{f})$ . Since  $f$  is expanding, we have that  $G$  is hyperbolic and there exists a non-empty  $G$ -forward invariant compact subset of  $F(G)$  by Proposition 2.8 (1). Hence, there exists  $R > 0$  such that, for each  $n \in \mathbb{N}$ , there exists a holomorphic branch  $\phi_n : B(f_{\omega_n}(z), R) \rightarrow \widehat{\mathbb{C}}$  of  $f_{\omega_n}^{-1}$  such that  $f_{\omega_n}(\phi_n(y)) = y$  for  $y \in B(f_{\omega_n}(z), R)$  and  $\phi_n(f_{\omega_n}(z)) = z$ . After making  $r_0$  sufficiently small, we may assume that, for the sets  $B_n$ , which are for  $n \in \mathbb{N}$  given by

$$B_n := \phi_n(B(f_{\omega_n}(z), r_0)),$$

we have that  $\text{diam}(f_{\omega_k}(B_n)) \leq r_0$  for all  $1 \leq k \leq n$ . Combining this with our assumption that  $M_\tau(\rho) = \rho$  and that  $\rho$  is constant on each connected component of  $F(G)$  by [Sum11, Theorem 3.15 (1)], we obtain that for all  $a, b \in B_n$ ,

$$(5.1) \quad |\rho(a) - \rho(b)| = p_{\omega_1} \cdots p_{\omega_n} \left| \rho(f_{\omega_n}(a)) - \rho(f_{\omega_n}(b)) \right|.$$

We set  $r_n := \|f'_{\omega_n}(z)\|^{-1}$  for each  $n \in \mathbb{N}$ . We may assume that  $(r_n)_{n \in \mathbb{N}}$  is strictly decreasing because  $f$  is expanding. Hence, for each  $r > 0$  sufficiently small, there exists a unique  $n \in \mathbb{N}$  such that  $r_{n+1} \leq r \leq r_n$ . To prove the lemma, our main task is to verify that there exists a constant  $C > 0$ , such that for all  $r > 0$  sufficiently small,

$$(5.2) \quad C^{-1} \leq \frac{\sup \{ |\rho(z) - \rho(y)| : y \in B(z, r) \}}{p_{\omega_1} \cdots p_{\omega_n}} \leq C.$$

To prove (5.2), we first observe that by Koebe's distortion theorem, there exist positive constants  $c_1, c_2$  such that for each  $n \in \mathbb{N}$ ,

$$B(z, c_1 r_n) \subset B_n \subset B(z, c_2 r_n).$$

Moreover, it is shown in [Sum11] that

$$J(G) = \left\{ y \in \widehat{\mathbb{C}} : \forall \varepsilon > 0 : \rho|_{B(y, \varepsilon)} \text{ is not constant} \right\}.$$

Since  $J(G)$  is compact, we see that there exists  $C_1 > 0$  such that for all  $y \in J(G)$ ,

$$(5.3) \quad \sup \{ |\rho(y) - \rho(y')| : y' \in B(y, r_0) \} \geq C_1.$$

Further, there exists  $k \in \mathbb{N}$  such that  $c_2 r_{n+k} \leq r_{n+1}$  and  $r_n \leq c_1 r_{n-k}$ , for all  $n$  sufficiently large. Consequently, for each  $r > 0$  we have

$$B(z, r) \supset B(z, r_{n+1}) \supset B(z, c_2 r_{n+k}) \supset B_{n+k} \quad \text{and} \quad B(z, r) \subset B(z, r_n) \subset B(z, c_1 r_{n-k}) \subset B_{n-k}.$$

Combining with (5.1) and (5.3) we obtain that

$$\begin{aligned} \sup \{ |\rho(z) - \rho(y)| : y \in B(z, r) \} &\geq \sup \{ |\rho(z) - \rho(y)| : y \in B_{n+k} \} \\ &= \sup \left\{ p_{\omega_1} \cdots p_{\omega_{n+k}} \left| \rho(f_{\omega_{n+k}}(z)) - \rho(f_{\omega_{n+k}}(y)) \right| : y \in B_{n+k} \right\} \\ &\geq p_{\omega_1} \cdots p_{\omega_{n+k}} C_1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sup \{ |\rho(z) - \rho(y)| : y \in B(z, r) \} &\leq \sup \{ |\rho(z) - \rho(y)| : y \in B_{n-k} \} \\ &= \sup \left\{ p_{\omega_1} \cdots p_{\omega_{n-k}} \left| \rho(f_{\omega_{n-k}}(z)) - \rho(f_{\omega_{n-k}}(y)) \right| : y \in B_{n-k} \right\} \\ &\leq p_{\omega_1} \cdots p_{\omega_{n-k}} 2 \max_{v \in C(\widehat{\mathbb{C}})} |\rho(v)|. \end{aligned}$$

We have thus proved (5.2).

By (5.2) and  $r_{n+1} \leq r \leq r_n$  we obtain that for each  $r > 0$ ,

$$\frac{S_n \tilde{\Psi}((\omega, z)) + \log C}{S_{n+1} \tilde{\zeta}((\omega, z))} \leq \frac{\log \sup \{ |\rho(z) - \rho(y)| : y \in B(z, r) \}}{\log r} \leq \frac{S_n \tilde{\Psi}((\omega, z)) - \log C}{S_n \tilde{\zeta}((\omega, z))}.$$

The lemma follows by letting  $r$  tend to zero.  $\square$

We are now in the position to state the main result of this section.

**Theorem 5.3.** *Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $f$  satisfies the separation condition. Let  $(p_i)_{i \in I} \in (0, 1)^I$  be a probability vector, let  $\tau := \sum_{i \in I} p_i \delta_{f_i}$  and suppose that there exists a non-constant function belonging to  $U_\tau$ . Let  $\rho \in C(\widehat{\mathbb{C}})$  be a non-constant function belonging to  $U_\tau$ . Let  $\psi = (\psi_i : f_i^{-1}(J(G)) \rightarrow \mathbb{R})_{i \in I}$  be given by  $\psi_i(z) := \log p_i$  and let  $t : \mathbb{R} \rightarrow \mathbb{R}$  denote the free energy function for  $(f, \psi)$ . Let  $\gamma$  be the unique number such that  $\mathcal{P}(\gamma \tilde{\psi}, \tilde{f}) = 0$ . Then we have the following.*

- (1) *There exists a number  $a \in (0, 1)$  such that  $\rho : \widehat{\mathbb{C}} \rightarrow \mathbb{C}$  is  $a$ -Hölder continuous and  $a \leq \alpha_-(\psi)$ .*
- (2) *We have  $\alpha_+(\psi) = \sup \{ \alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset \}$  and  $\alpha_-(\psi) = \inf \{ \alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset \}$ . Moreover,  $H$  can be replaced by  $R_*$ ,  $R$  or  $R^*$ .*
- (3) *Let  $\alpha_\pm := \alpha_\pm(\psi)$  and  $\alpha_0 := \alpha_0(\psi)$ . If  $\alpha_- < \alpha_+$  then we have for each  $\alpha \in (\alpha_-, \alpha_+)$ ,*

$$\begin{aligned} \dim_H(\mathcal{F}(\alpha, \psi)) &= \dim_H(\mathcal{F}^\sharp(\alpha, \psi)) = \dim_H(R^*(\rho, \alpha)) = \dim_H(R_*(\rho, \alpha)) \\ &= \dim_H(R(\rho, \alpha)) = \dim_H(H(\rho, \alpha)) = -t^*(-\alpha) > 0. \end{aligned}$$

*Moreover,  $s(\alpha) := -t^*(-\alpha)$  is a real analytic strictly concave positive function on  $(\alpha_-, \alpha_+)$  with maximum value  $\delta = -t^*(-\alpha_0) > 0$ . Also,  $s'' < 0$  on  $(\alpha_-, \alpha_+)$ .*

- (4) (a) For each  $i \in I$  we have  $\deg(f_i) \geq 2$ . Moreover, we have  $\alpha_- = \alpha_+$  if and only if there exist an automorphism  $\varphi \in \text{Aut}(\widehat{\mathbb{C}})$  and  $(a_i) \in \mathbb{C}^I$  such that

$$\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)} \quad \text{and} \quad \log \deg(f_i) = -(\gamma/\delta) \log p_i.$$

- (b) If  $\alpha_- = \alpha_+$  then we have

$$\mathcal{F}(\alpha_0, \psi) = \mathcal{F}^\sharp(\alpha_0, \psi) = R^*(\rho, \alpha_0) = R_*(\rho, \alpha_0) = R(\rho, \alpha_0) = H(\rho, \alpha_0) = J(G),$$

where  $\dim_H(J(G)) = \delta > 0$ , and for all  $\alpha \neq \alpha_0$  we have

$$\mathcal{F}(\alpha, \psi) = \mathcal{F}^\sharp(\alpha, \psi) = R^*(\rho, \alpha) = R_*(\rho, \alpha) = R(\rho, \alpha) = H(\rho, \alpha) = \emptyset.$$

*Proof.* By the separation condition and  $J(G) = \bigcup_{i \in I} f_i^{-1}(J(G))$  (see [Sum00]) we have that the kernel Julia set  $J_{\ker}(G) := \bigcap_{g \in G} g^{-1}(J(G))$  of  $G$  is empty. From this and the assumption that there exists a non-constant unitary eigenfunction of  $M_\tau$  in  $C(\widehat{\mathbb{C}})$  ([Sum11, Theorem 3.15 (21), Remark 3.18]), it follows that  $\deg(f_i) \geq 2$  for each  $i \in I$ . Moreover, by [Sum13, Theorem 3.29] there exists a constant  $a \in (0, 1)$  such that  $\rho : \widehat{\mathbb{C}} \rightarrow \mathbb{C}$  is  $a$ -Hölder continuous.

Since  $f$  satisfies the separation condition, by passing to  $(f_\omega)_{\omega \in I^k}$  where  $k$  is a sufficiently large positive integer, we may assume that  $f$  satisfies the open set condition. Also note that there exists  $\gamma \in \mathbb{R}$  such that  $\mathcal{P}(\gamma\tilde{\psi}, \tilde{f}) = 0$ .

Let  $\alpha \in \mathbb{R}$ . Recall that  $H(\rho, \alpha) = R_*(\rho, \alpha)$  by Lemma 5.1. We only give the proof of (2), (3) and (4) for the level set  $R_*(\rho, \alpha)$ . The sets  $R^*(\rho, \alpha)$  and  $R(\rho, \alpha)$  can be considered in a similar fashion. The main task is to show that  $\mathcal{F}(\alpha, \psi) \subset R_*(\rho, \alpha) \subset \mathcal{F}^\sharp(\alpha, \psi)$ . Then the assertion in (2) follows from Theorem 4.5 (1) and Lemma 4.3 (2), and the assertions in (3) follow from Theorem 4.5 (1), (2) and (3). The assertion in (4a) follows from Proposition 4.6. To prove (4b) we observe that by the proof of Proposition 4.6 there exists a continuous function  $\tilde{h} : J(\tilde{f}) \rightarrow \mathbb{R}$  such that

$$\frac{S_n \tilde{\psi}(x)}{S_n \tilde{\zeta}(x)} = \frac{\delta S_n \tilde{\psi}(x)}{\gamma S_n \tilde{\psi}(x) + \tilde{h}(x) - \tilde{h} \circ \tilde{f}^{n+1}(x)}, \quad \text{for every } x \in J(\tilde{f}),$$

which shows that  $\lim_{n \rightarrow \infty} S_n \tilde{\psi}(x) / S_n \tilde{\zeta}(x) = \delta / \gamma$  for every  $x \in J(\tilde{f})$ . Now (4b) follows from Theorem 4.5 (1) and (3) and Lemma 4.3 (2). Finally, by combining with the fact that  $\rho$  is  $a$ -Hölder continuous, we obtain that  $a \leq \alpha_-(\psi)$ .

To complete the proof, we verify that  $\mathcal{F}(\alpha, \psi) \subset R_*(\rho, \alpha) \subset \mathcal{F}^\sharp(\alpha, \psi)$ . To prove that  $\mathcal{F}(\alpha, \psi) \subset R_*(\rho, \alpha)$ , let  $z \in \mathcal{F}(\alpha, \psi)$ . By definition of  $\mathcal{F}(\alpha, \psi)$ , there exists  $\omega \in I^\mathbb{N}$  such that  $(\omega, z) \in \tilde{\mathcal{F}}(\alpha, \psi)$ . Hence, by Lemma 5.2, we have that  $Q(\rho, z) = \alpha$  and thus,  $z \in R(\rho, \alpha) \subset R_*(\rho, \alpha)$ . To verify that  $R_*(\rho, \alpha) \subset \mathcal{F}^\sharp(\alpha, \psi)$ , let  $z \in R_*(\rho, \alpha)$ , that is,  $Q_*(\rho, z) = \alpha$ . Since  $\rho$  is constant on each connected component of  $F(G)$  by [Sum11, Theorem 3.15 (1)], we have that  $z \in J(G)$ . By Proposition 2.1 (3), there exists  $\omega \in I^\mathbb{N}$  such that  $(\omega, z) \in J(\tilde{f})$ . Lemma 5.2 gives that  $\liminf_n S_n \tilde{\psi}((\omega, z)) / S_n \tilde{\zeta}((\omega, z)) = \alpha$ , which implies that  $(\omega, z) \in \tilde{\mathcal{F}}^\sharp(\alpha, \psi)$ . Hence, we have  $z \in \mathcal{F}^\sharp(\alpha, \psi)$ . We have thus shown that  $\mathcal{F}(\alpha, \psi) \subset R_*(\rho, \alpha) \subset \mathcal{F}^\sharp(\alpha, \psi)$  and the proof is complete.  $\square$

## 6. EXAMPLES

We give some examples to which we can apply the main theorems.

- (1) Let  $f = (f_i)_{i \in I} \in (\text{Rat})^I$  be expanding and let  $G = \langle f_i : i \in I \rangle$ . Suppose that  $f$  satisfies the separation condition. Let  $(p_i)_{i \in I} \in (0, 1)^I$  be a probability vector, let  $\tau := \sum_{i \in I} p_i \delta_{f_i}$ . Suppose that  $G$  has at least two minimal sets. Here, we say that a non-empty compact subset  $L$  of  $\widehat{\mathbb{C}}$  is a minimal

set of  $G$  if  $L$  is minimal among the space  $\{K \mid K \text{ is a non-empty compact subset of } \widehat{\mathbb{C}} \text{ and } g(K) \subset K \text{ for each } g \in G\}$  with respect to the inclusion. (Note that if  $K$  is a non-empty compact subset  $\widehat{\mathbb{C}}$  such that  $g(K) \subset K$  for each  $g \in G$ , then by Zorn's lemma, there exists a minimal set  $L$  of  $G$  with  $L \subset K$ .) Let  $T_{L,\tau} : \widehat{\mathbb{C}} \rightarrow [0, 1]$  be the function of probability of tending to  $L$  which is defined as

$$T_{L,\tau}(z) = \left( \otimes_{n=1}^{\infty} \tau \right) \left( \left\{ \omega = (\omega_1, \omega_2, \dots) \in \{f_i \mid i \in I\}^{\mathbb{N}} \mid d(\omega_n \cdots \omega_1(z), L) \rightarrow 0 \text{ as } n \rightarrow \infty \right\} \right).$$

Then by [Sum11]  $T_{L,\tau}$  is a non-constant function belonging to  $U_{\tau}$ . In fact,  $M_{\tau}(T_{L,\tau}) = T_{L,\tau}$ . Thus, we can apply Theorem 5.3 to  $f$  and  $\rho = T_{L,\tau}$ . The function  $T_{L,\tau}$  can be regarded as a complex analogue of the devil's staircase and Lebesgue's singular functions (see [Sum11, Introduction]). If each  $f_i$  is a polynomial and  $L = \{\infty\}$  then  $T_{\infty,\tau} := T_{\{\infty\},\tau}$  is sometimes called a devil's coliseum.

Since  $(f_{\omega})_{\omega \in I^k}$ , where  $k$  is a large positive integer, satisfies the open set condition, all statements in Theorem 4.5 and Proposition 4.6 hold for  $f$ .

- (2) Let  $f_1, f_2$  be two polynomials with  $\deg(f_i) \geq 2$  for  $i \in \{1, 2\}$ . Let  $I = \{1, 2\}$  and  $G = \langle f_i : i \in I \rangle$ . Let  $(p_1, p_2) \in (0, 1)^2$  with  $p_1 + p_2 = 1$  and let  $\tau = p_1 \delta_{f_1} + p_2 \delta_{f_2}$ . Suppose that  $G$  is hyperbolic,  $P(G) \setminus \{\infty\}$  is bounded in  $\mathbb{C}$  and that  $J(G)$  is disconnected. Then  $f = (f_1, f_2)$  is expanding,  $f$  satisfies the separating condition (see [Sum09, Theorem 1.7]) and the function of probability of tending to infinity  $T_{\infty,\tau} = T_{\{\infty\},\tau} : \widehat{\mathbb{C}} \rightarrow [0, 1]$  is a non-constant function belonging to  $U_{\tau}$ . To this  $f$  and  $\rho = T_{\infty,\tau}$  we can apply Theorem 5.3. To see a concrete example, let  $g_1(z) = z^2 - 1, g_2(z) = z^2/4, f_1 = g_1 \circ g_1, f_2 = g_2 \circ g_2$ . Then  $f = (f_1, f_2)$  satisfies the above condition (see [Sum11, Example 6.2]).
- (3) Let  $f_1$  be a hyperbolic polynomial with  $\deg(f_1) \geq 2$ . Suppose that  $J(f_1)$  is connected. Let  $h \in \text{Int}(K(f_1))$ . Let  $d \in \mathbb{N}$  with  $d \geq 2$  and  $(\deg f_1, d) \neq (2, 2)$ . Then there exists a constant  $a_0 > 0$  such that for each  $a \in \mathbb{C}$  with  $0 < |a| < a_0$ , setting  $f_{2,a}(z) := a(z-h)^d + h$ , we have that (i)  $G_a := \langle f_1, f_{2,a} \rangle$  is hyperbolic, (ii)  $f_a = (f_1, f_{2,a})$  satisfies the separating condition and (iii)  $P(\langle f_1, f_{2,a} \rangle) \setminus \{\infty\}$  is bounded in  $\mathbb{C}$  (see [Sum11, Proposition 6.1]).
- (4) There are many examples of  $f = (f_i)_{i \in I} \in \text{Rat}^I$ , which satisfy the assumptions of Theorem 5.3 (see [Sum11, Propositions 6.3, 6.4 and 6.5]).

Finally we give an important remark on the estimate of  $\alpha_-$  and the non-differentiability of non-constant  $\rho \in U_{\tau}$ .

*Remark 6.1.* Let  $f = (f_i)_{i \in I} \in \text{Rat}^I$  and suppose that each  $f_i$  is a polynomial with  $\deg(f_i) \geq 2$ . Under the assumptions of Theorem 5.3, we have by Theorem 5.3 and [Sum11, Theorem 3.82] that

$$\alpha_-(\psi) \leq \frac{-\sum_{i \in I} p_i \log p_i}{\sum_{i \in I} p_i \log \deg f_i + \int_{\Gamma^{\mathbb{N}}} \sum_c \mathcal{G}_{\gamma}(c) d\widehat{\tau}(\gamma)} \leq \alpha_+(\psi),$$

where  $\Gamma = \{f_i : i \in I\}$ ,  $\widehat{\tau} = \otimes_{j=1}^{\infty} \tau$ , and  $\mathcal{G}_{\gamma}$  denotes the Green's function of the basin  $A_{\infty,\gamma}$  of  $\infty$  for the sequence  $\gamma = (\gamma_1, \gamma_2, \dots) \in \Gamma^{\mathbb{N}}$  and  $c$  runs over the critical points of  $\gamma_1$  in  $A_{\infty,\gamma}$ . For the details we refer to [Sum11, Theorem 3.82].

Moreover, in addition to the assumptions of Theorem 5.3, if each  $f_i$  is a polynomial with  $\deg(f_i) \geq 2$  and if (a)  $\sum_{i \in I} p_i \log(p_i \deg f_i) > 0$  or (b)  $P(G) \setminus \{\infty\}$  is bounded in  $\mathbb{C}$  or (c)  $\text{card}(I) = 2$ , then

$$\alpha_-(\psi) \leq \frac{-\sum_{i \in I} p_i \log p_i}{\sum_{i \in I} p_i \log \deg f_i + \int_{\Gamma^{\mathbb{N}}} \sum_c \mathcal{G}_{\gamma}(c) d\widehat{\tau}(\gamma)} < 1.$$

See [Sum11, Theorem 3.82]. For the proof, we use potential theory.

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